<u>Recall</u>: • $\mathrm{Pic}^0(X) := \mathrm{Div}_0(X) / \text{linearly equiv.}$ deg. 0 divisors modulo linearly equiv.

• $\mathrm{Jac}(X) := \dfrac{H^0(X,\Omega^1)^\vee}{\iota(H_1(X,\mathbb{Z}))}$ where: $\iota : H_1(X,\mathbb{Z}) \longrightarrow H^0(X,\Omega^1)^\vee$
$[\gamma] \longmapsto (\omega \mapsto \int_\gamma \omega)$.

holo. 1-forms modulo period integrals

• Fix a base point $p_0 \in X$ ⇝ <u>Abel-Jacobi map</u>.

$$AJ : X \longrightarrow \mathrm{Jac}(X)$$
$$p \longmapsto \left[\left(\omega \mapsto \int_{p_0}^{p} \omega\right)\right]$$

If we change the path, the integrals changed by a period.

• Extend to: $AJ : \mathrm{Div}_0(X) \longrightarrow \mathrm{Jac}(X)$ gp homom.

$$\sum n_p P \longmapsto \left[\omega \mapsto \sum n_p \int_{p_0}^{P} \omega\right]$$

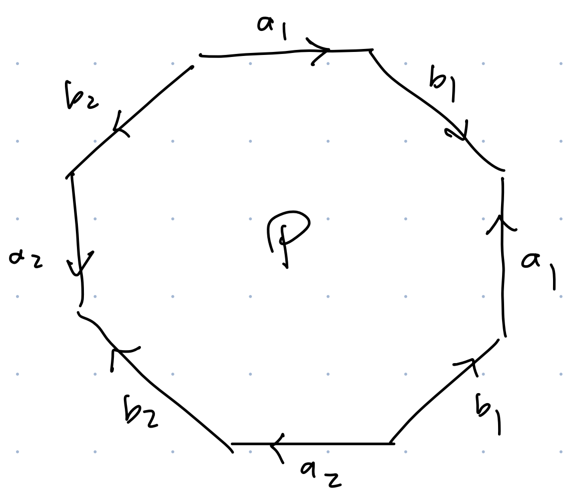
<u>Rmk</u>: AJ on $\mathrm{Div}_0(X)$ is indep. of the choice of basept p_0:

$$\sum n_p \int_{p_0}^{P} \omega - \sum n_p \int_{Q_0}^{P} \omega = \sum n_p \int_{p_0}^{Q_0} \omega = 0 \text{ since } \underline{\underline{\sum n_p = 0}}.$$

<u>Thm</u> (Abel-Jacobi) $AJ : \mathrm{Div}_0(X) / \text{linear equiv} \xrightarrow{\sim} \mathrm{Jac}(X)$.

i.e. <u>Abel</u>: For $D \in \mathrm{Div}_0(X)$, $AJ(D) = 0 \iff D = (f)$.

<u>Jacobi</u>: Every elt of $\mathrm{Jac}(X)$ is of the form $AJ(D)$, $D \in \mathrm{Div}_0(X)$.



$g=2$ example.

$H_1(X,\mathbb{Z}) \cong \mathbb{Z}^4 \cong \langle a_1, b_1, a_2, b_2 \rangle$.

<u>Riemann bilinear relation:</u> α, β smooth closed 1-forms on X.

$$\int_X \alpha\wedge\beta = \sum_{i=1}^{g} \left(\int_{a_i}\alpha \int_{b_i}\beta - \int_{b_i}\alpha \int_{a_i}\beta \right)$$

pf: • $\exists$ F smooth fcn on the polygon P st. $dF=\alpha$.
(since P is simply connected.)

$\Longrightarrow$ $\alpha\wedge\beta = dF\wedge\beta = d(F\beta)$ since $d\beta=0$.

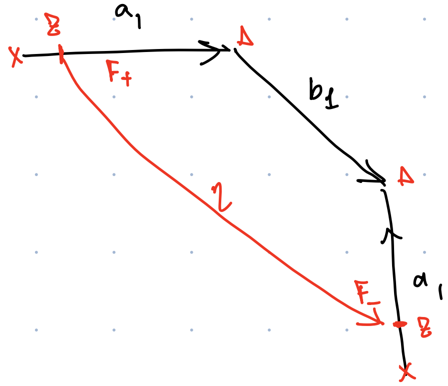
$\Longrightarrow$ $\displaystyle\int_X \alpha\wedge\beta = \int_P \alpha\wedge\beta = \int_{\partial P} F\beta.$

$\uparrow$

$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots$

• Let F_+ be the value of F on a_i,
and F_- —— " —————— on a_i^{-1}.

Then $\displaystyle\int_{a_i} F\beta + \int_{a_i^{-1}} F\beta = \int_{a_i} (F_+ - F_-)\beta.$

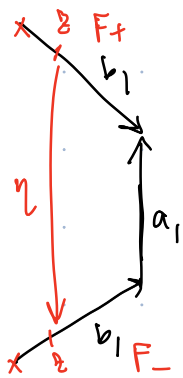


$$F_+(z) - F_-(z) = -\int_\eta \alpha = -\int_{b_1} \alpha$$

$$\eta \sim b_1$$

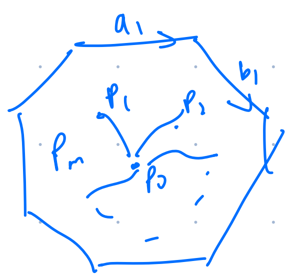
$$\Rightarrow \int_{a_i} \overbrace{(F_+ - F_-)}^{\text{const. on } a_i} \beta = \left(-\int_{b_i} \alpha\right) \int_{a_i} \beta.$$

Similarly,



$$\int_{b_i} F\beta + \int_{b_i^{-1}} F\beta = \int_{b_i} (F_+ - F_-)\beta$$

$$= \left(\int_{a_i} \alpha\right)\left(\int_{b_i} \beta\right).$$



basept. $p_0 \notin \{p_1, \dots, p_m\}$

Another version: ω - holo. 1-form.

η - mero. 1-form w/ simple poles at $p_1, \dots, p_m$.

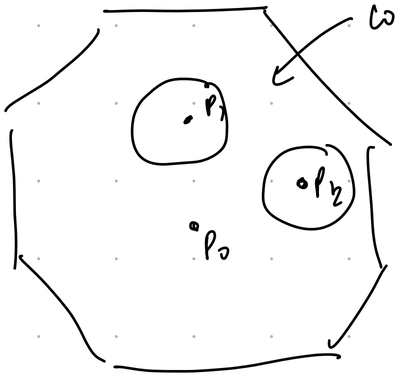
w/ residues $\mathrm{res}_{p_j}(\eta) = n_j$. $\left(\sum n_j = 0\right)$.

Then:

$$2\pi i \sum_j n_j \int_{p_0}^{p_j} \omega \equiv \sum_{i=1}^{g} \left(\int_{a_i} \omega \int_{b_i} \eta - \int_{b_i} \omega \int_{a_i} \eta \right)$$

up to periods of $\omega \cdot 2\pi i$

pf:



ω, η holo. $\quad dF = \omega$ in the polygon.

$$d(F\eta) = \underbrace{\omega \wedge \eta}_{=0} + F \wedge \underbrace{d\eta}_{=0}$$

away from pole (both holo. 1-form) $\qquad$ away from pole

$$\xrightarrow{\text{Stokes}} \int_{\partial P} F\eta = \sum_j \int_{\partial D_j(\varepsilon)} F\eta \xrightarrow{\varepsilon \to 0} 2\pi i\, F(p_j)\, n_j$$

$$= 2\pi i \left(\int_{p_0}^{p_j} \omega \right) n_j$$

$\square$

Claim: $AJ(\underset{\sum_p \mathrm{ord}_p(f)\cdot p}{\overset{D}{(f)}}) = \left[\omega \mapsto \sum_p \mathrm{ord}_p(f) \cdot \int_{p_0}^{p} \omega \right] \equiv 0.$ in $Jac(X)$

i.e. $\displaystyle \sum_p \underbrace{\mathrm{ord}_p(f)}_{= \mathrm{res}_p\left(\frac{df}{f}\right)} \cdot \int_{p_0}^{p} \omega \equiv 0 \left(\text{up to periods of } \omega \right)$ $\quad \forall$ holo. 1-form ω.

Apply the bilinear relation above:

$$2\pi i \sum_p \mathrm{res}_p \Big(\underbrace{\frac{df}{f}}_{=\eta} \Big) \int_{p_0}^{p} \omega \equiv \sum_{i=1}^{g} \left(\int_{a_i} \omega \int_{b_i} \frac{df}{f} - \int_{b_i} \omega \int_{a_i} \frac{df}{f} \right)$$

$$\int_{\text{closed cycle}} \frac{df}{f} \in 2\pi i \mathbb{Z}.$$

$|||$

0 up to $2\pi i$ (periods of ω). . $\square$

Claim: For $D = \sum n_j P_j$ with $\sum n_j = 0$.

If $AJ(D) = 0$, then $\exists f \in \mathcal{M}(X)$ w/ $\mathrm{ord}_{P_j}(f) = n_j$.

Idea: If f is such a mero fcn, then $\eta = \frac{df}{f}$ is a mero. 1-form

s.t. 1) $\mathrm{res}_{P_j}(\eta) = n_j$

2) $\int_{\substack{\text{closed} \\ \text{cycle}}} \eta \in 2\pi i \mathbb{Z}$.

and we can recover f from η by integrating + expotentiating.

Strategy: Construct such mero. 1-form η, $\xrightarrow{\exp(\int \eta)}$ obtain f.

Steps: 1) Find a mero. 1-form η_0 w/ simple poles at $P_1, \dots, P_m$
and residues $n_1, \dots, n_m$.

(exists by M-L problem for 1-forms. since $\sum n_j = 0$!)

2) [Such η_0 may not satisfy $\int_{\substack{\text{closed} \\ \text{cycle}}} \eta_0 \in 2\pi i \mathbb{Z}$]

Find holo. 1-form α s.t.

$$\int_\gamma (\eta_0 + \alpha) \in 2\pi i \mathbb{Z} \quad \forall \gamma \in H_1(X, \mathbb{Z}).$$

[This step requires $AJ(D) = 0$]

It remains to show Step 2.

Recall: $$\underbrace{2\pi i \sum_p n_p \int_{p_0}^{p} \omega}_{} = \sum_{i=1}^{g} \Big(\int_{a_i} \omega \underbrace{\int_{b_i} \eta_0}_{\overset{B_i}{\shortparallel}} - \int_{b_i} \omega \underbrace{\int_{a_i} \eta_0}_{\overset{A_i}{\shortparallel}} \Big)$$

holo

$$\shortparallel$$

$$2\pi i \cdot \underset{\overset{\shortparallel}{0}}{AJ(D)}(\omega) \equiv 2\pi i \,(\text{periods of } \omega)$$ since $\{a_1, b_1, \dots\}$ is basis of $H_1(X, \mathbb{Z})$

$$\shortparallel$$

$$2\pi i \Big(\sum_i m_i \int_{a_i} \omega + \sum_i n_i \int_{b_i} \omega \Big)$$

$$\Rightarrow \sum_i (B_i - 2\pi i m_i) \int_{a_i} \omega + \sum_i (-A_i - 2\pi i n_i) \int_{b_i} \omega = 0$$

$\forall\, \omega$ – holo. 1-form.

Want: Find holo. 1-form α s.t. $\underset{\overset{\shortparallel}{A_i + \int_{a_i} \alpha}}{\int_{a_i} (\eta_0 + \alpha)}$, $\underset{\overset{\shortparallel}{B_i + \int_{b_i} \alpha}}{\int_{b_i} (\eta_0 + \alpha)} \in 2\pi i \mathbb{Z}$.

Question: Given $x_i, y_i \in \mathbb{C}$, When does there exist α – holo 1-form w/ $\int_{a_i} \alpha = x_i$, $\int_{b_i} \alpha = y_i$?

Need: $\forall\, \omega$, Riemann bilinear $\Rightarrow$

$$\sum_i \Big(x_i \int_{b_i} \omega - y_i \int_{a_i} \omega \Big) = 0.$$

Conversely, suppose x_i, y_i satisfies

$$\sum_i \left(x_i \int_{b_i} \omega - y_i \int_{a_i} \omega \right) = 0. \quad \forall \omega.$$

<u>Claim</u>: Such α exists:

- $H^0(X, \mathcal{O}^1) \xrightarrow{\cong} \mathbb{C}^g$

$$\alpha \longmapsto \left(\int_{a_1} \alpha, \cdots, \int_{a_g} \alpha \right)$$

$\Rightarrow \exists \alpha$ holo. 1-form w/ $\int_{a_i} \alpha = x_i$.

- Suppose $\int_{b_i} \alpha = y_i + \delta_i$.
- Riemann bilinear $\Rightarrow \sum_i \left(x_i \int_{b_i} \omega - (y_i + \delta_i) \int_{a_i} \omega \right) = 0 \ \forall \omega$

$$\Rightarrow \sum_i \delta_i \int_{a_i} \omega = 0 \quad \forall \omega.$$

$$\Rightarrow \delta_i \equiv 0.$$

$\Rightarrow \exists \alpha$ - holo. 1-form s.t.

$$\int_{a_i} \alpha = -A_i - 2\pi i n_i, \quad \int_{b_i} \alpha = -B_i + 2\pi i m_i$$

$$\Rightarrow \quad A_i + \int_{a_i} \alpha, \quad B_i + \int_{b_i} \alpha \in 2\pi i \mathbb{Z}. \quad \square$$

Claim: $\mathrm{Div}^0(X) \xrightarrow{AJ} \mathrm{Jac}(X)$ is surjective.

Actually, we'll prove: $X^g \xrightarrow{\Phi} \mathrm{Jac}(X)$ is surjective.

$$(P_1, \cdots, P_g) \longmapsto AJ(P_1 + \cdots + P_g - gP_0)$$

$$\Phi(P_1, \cdots, P_g)(\omega) = \sum_{i=1}^{g} \int_{P_0}^{P_i} \omega \quad \text{mod. periods.}$$

- Both X^g, $\mathrm{Jac}(X)$ cpt. $\Rightarrow$ Φ is proper holo.

  $\Rightarrow$ $\Phi(X^g) \subseteq \mathrm{Jac}(X)$ is closed analytic subset.

$\Rightarrow$ To show $\Phi(X^g) = \mathrm{Jac}(X)$, it suffices to show $\Phi(X^g)$ contains a nonempty open subset of $\mathrm{Jac}(X)$.

- It suffices to show that $d\Phi|_{(P_1, \cdots, P_g)}$ is an isom. at some pt. $(P_1, \cdots, P_g) \in X^g$.

- $d\Phi|_{(P_1, \cdots, P_g)} : T_{(P_1, \cdots, P_g)} X^g \longrightarrow T_{pt} \mathrm{Jac}(X)$

$$T_{(P_1, \cdots, P_g)} X^g \cong \bigoplus_i T_{P_i} X \qquad \left(\mathrm{Jac}(X) = \frac{H^0(X, \Omega^1)^\vee}{\iota(H_1(X, \mathbb{Z}))}, \quad T_{pt} \mathrm{Jac}(X) \cong H^0(X, \Omega^1)^\vee \right)$$

$$(v_1, \cdots, v_g) \longmapsto \left(\omega \mapsto \sum_i \omega_{P_i}(v_i) \right)$$

- $0 \longrightarrow K_X(-E) \longrightarrow K_X \xrightarrow{ev} K_X|_E \longrightarrow 0$, $\quad E = P_1 + \cdots + P_g$

  $H^0(X, K_X) \xrightarrow{ev} H^0(X, K_X|_E)$ is dual to $d\Phi|_{(P_1, \cdots, P_g)}$

  both g-dim'l.

- Suffices to show: $H^0(X, K_X(-E)) = 0$.

<u>Claim:</u> $\exists\ p_1, \dots, p_g \in X$ s.t. no holo. 1-form vanishes at all of them.

$\left(\Rightarrow\ H^0(X, K_X(-E)) = 0.\right)$

<u>pf:</u> Prove by induction. Start with $V_0 = H^0(X, K_X) \to g$-dim'l.

- Suppose we've chosen $p_1, \dots, p_r$ s.t.

$$V_r = H^0(X, K_X(-p_1 - \cdots - p_r)) \text{ has dim } g-r.$$

- If $r < g$, then $V_r \neq 0$. Choose $\omega \in V_r \setminus \{0\}$.
- Choose p_{r+1} s.t. ω not vanish at p_{r+1}.

$$\Rightarrow\ 0 \to K_X(-p_1 - \cdots - p_r - p_{r+1}) \to K_X(-p_1 - \cdots - p_r) \xrightarrow{ev} K_X(-p_1 - \cdots - p_r)|_{p_{r+1}} \to 0$$

$$\Rightarrow\ 0 \to H^0(K_X(-p_1 - \cdots - p_{r+1})) \to H^0(K_X(-p_1 - \cdots - p_r)) \xrightarrow{\text{surjective}} \mathbb{C}$$

$$\omega \longmapsto \neq 0$$

$$\Rightarrow\ \dim H^0(K_X(-p_1 - \cdots - p_{r+1})) = \dim H^0(K_X(-p_1 - \cdots - p_r)) - 1. \qquad \square$$

<u>Algebraic proof of Abel-Jacobi thm.</u>

Consider $0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0.$

$$\Rightarrow\ \begin{tikzcd}
H^1(X, \underline{\mathbb{Z}}_X) \arrow[r] \arrow[d, "\wr"', "\text{Poincaré duality}"] \arrow[dr, phantom, "\text{HARD EX}"] & H^1(X, \mathcal{O}_X) \arrow[r] \arrow[d, "\wr\ \text{Serre}"] & H^1(X, \mathcal{O}_X^\times) \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\text{HARD EX}"] & H^2(X, \underline{\mathbb{Z}}_X). \arrow[d, "\wr\ \text{Poincaré duality}"] \\
H_1(X, \mathbb{Z}) \arrow[r, "\iota"] & H^0(X, \Omega^1_X)^\vee & \mathrm{Pic}(X) \arrow[r, "\deg"] & \mathbb{Z}
\end{tikzcd}$$

$\square$