Recap:

$(f)+D \geq 0$

- Riemann-Roch: D - divisor. $\rightsquigarrow$ $\mathcal{O}_D$

  $$h^0(\mathcal{O}_D) - h^1(\mathcal{O}_D) = 1 - g + \deg(D).$$

- Serre duality: $H^1(X, \mathcal{O}_D) \cong H^0(X, \mathcal{O}_{K-D})^*$ $\quad$ K - canonical div. on X.

D, $\deg(D) < 0$
$f \in H^0(X, \mathcal{O}_D)$
$\underline{(f)+D} \geq 0$

Coro: If $\deg(D) > 2g-2$, then $h^1(\mathcal{O}_D) = 0$.

(Since: $H^1(X, \mathcal{O}_D) \cong H^0(X, \mathcal{O}_{K-D})^* = 0.$)

$\deg(K-D) = (2g-2) - \deg(D) < 0.$

$(\Rightarrow h^0(D) = 1 - g + \deg(D).)$

Coro: $\mathcal{M}$ - sheaf of mero. fcns. $\quad H^1(X, \mathcal{M}) = 0.$

pf: Let (f_{ij}) be a cocycle of a class in $H^1(X, \mathcal{M})$.

f_{ij}: mero. fcn on $\mathcal{U}_{ij}$.

$\exists D$ of $\deg D > 2g-2$. s.t. $\underline{(f_{ij})}$ is a cocycle of a class in $H^1(X, \mathcal{O}_D)$

By Coro $\Rightarrow$ $\underline{H^1(X, \mathcal{O}_D) = 0}$ $\Rightarrow$ (f_{ij}) is a coboundary

i.e. $\exists (g_i) \in \mathcal{C}^0(\mathcal{U}, \mathcal{O}_D)$ s.t. $\underline{\underline{f_{ij} = g_i - g_j}}$

$$\mathcal{C}^0(\mathcal{U}, \mathcal{O}_D) \subset \mathcal{C}^0(\mathcal{U}, \mathcal{M})$$

$\Rightarrow$ (f_{ij}) is a coboundary in $\mathcal{C}^1(\mathcal{U}, \mathcal{M})$.

$\Rightarrow$ $[(f_{ij})] = 0$ in $H^1(X, \mathcal{M})$. $\square$

§ embedding of Riemann surfaces into $\mathbb{CP}^N$

Def. D div. on a cpt. R.S. X. . The <u>complete linear system</u>

$$|D| := \{ E \mid \underset{\substack{\text{linearly}\\ \text{equiv.}}}{E \sim D},\ \underset{\text{effective}}{E \geq 0} \}$$

$(E - D = (f))$

$$= \{ E \mid E = D + (f),\ f \in \boxed{H^0(X, \mathcal{O}_D)} \}$$

Rmk:

- Replace (f) by (cf) for some $c \in \mathbb{C}^\times$, $\rightsquigarrow (f) = (cf)$.
- If $(f) = (g)$, then $f = cg$ for some $c \in \mathbb{C}^\times$.
  ($\Rightarrow fg^{-1}$ has no zero/pole. $\Rightarrow$ const.)

$\Rightarrow \quad |D| \cong \mathbb{P}H^0(X, \mathcal{O}_D)$.

$\dim |D| = h^0(\mathcal{O}_D) - 1$.

- If $D \sim D'$, then $|D| = |D'|$.
- All divisors in $|D|$ has the same deg., so $\deg |D| := \deg(D)$

Def: A point $p \in X$ is a <u>basepoint of D (or $|D|$)</u> if
$$p \leq E \quad \forall E \in |D|$$

Say $|D|$ is <u>basepoint free</u> if it has no basepoints.

e.g. $X = \mathbb{C}/\Lambda$, $p_0 \in X$, $\boxed{D := p_0}$

$$|D| = \{ E \mid E = \underset{\substack{\| \\ p_0}}{D} + (f) \text{ effective} \} = \{p_0\}.$$

↑ mero. fcn, has at worst a simple pole at p_0

$\Downarrow$

f holo. $\Rightarrow$ f const.

$\Rightarrow$ p_0 is a basepoint of D. (D is not basepoint free)

e.g. $X = \mathbb{C}/\Lambda$, $D = 2p_0$.

$$|D| = \{ E \mid E = \underset{\substack{\| \\ 2p_0}}{D} + (f) \text{ eff.} \}$$

↑ mero. fcn has at worst double pole at p_0

- $\boxed{2p_0} \in |D|$
- $f = \wp(z - p_0)$ has two zeros $q_1, q_2 \neq p_0$. double pole at p_0

$$D + (f) = \boxed{q_1 + q_2} \in |D|$$

$\Rightarrow$ $|D|$ is basepoint free.

Rmk: • When p is a basepoint of D, we have

$$|D| = |D-p| + p \quad \text{and} \quad H^0(X, \mathcal{O}_D) = H^0(X, \mathcal{O}_{D-p})$$

$$\boxed{(f)+D} - p$$

• If p is not a basepoint of D, then

$$h^0(\mathcal{O}_D) = h^0(\mathcal{O}_{D-p}) + 1.$$

$$\Big(0 \to \mathcal{O}_{D-p} \to \mathcal{O}_D \to \mathbb{C}_p \to 0$$

skyscraper sheaf at p.

$$\rightsquigarrow \quad 0 \to H^0(\mathcal{O}_{D-p}) \to H^0(\mathcal{O}_D) \to \boxed{\mathbb{C}}$$
$$\to H^1(\mathcal{O}_{D-p}) \to \cdots \Big)$$

• p is a basept of $D \iff \forall f \in H^0(\mathcal{O}_D),\ p \in D + (f)$.

$\iff \operatorname{ord}_p(f) > -D(p) \quad \forall f \in H^0(\mathcal{O}_D)$.

p is not a basept. of $D \iff \exists\, \boxed{f \in H^0(\mathcal{O}_D)}$ s.t. $\boxed{\operatorname{ord}_p(f) = -D(p)}$

$\iff \exists f \in H^0(\mathcal{O}_D)$ s.t. $\underbrace{\mathcal{O}_{D,p}}_{} = \underbrace{\mathcal{O}_p \cdot (f)}_{}$

$p \leftarrow z$ $\quad D(p) = n$ $\quad$ stalk of $\mathcal{O}_D$ at p $\quad$ $\mathcal{O}_p$-module generated by f.

$\{ a_{-n} z^{-n} + a_{-(n-1)} z^{-(n-1)} + \cdots \}$, $\quad \{a_0 + a_1 z + \cdots\}$ $\quad b_{-n} z^{-n} + \cdots$, $\neq 0$

Def: $\mathcal{O}_D$ is generated by global section if

$\forall x \in X$, $\exists f \in H^0(X, \mathcal{O}_D)$ s.t. $\mathcal{O}_{D,x}$ is generated by $f|_x$

as an $\mathcal{O}_x$-module. i.e. $\mathcal{O}_{D,x} = \mathcal{O}_x \cdot f|_x$.

The following are equivalent:

- $|D|$ is basepoint free
- $\boxed{h^0(\mathcal{O}_D) = h^0(\mathcal{O}_{D-p}) + 1 \quad \forall p \in X.}$
- $\mathcal{O}_D$ is generated by global section
- $\forall p \in X, \ \exists f \in H^0(\mathcal{O}_D)$ s.t. $\mathrm{ord}_p(f) = -D(p)$.

Rmk:

- X cpt RS genus g. $\quad P \in X, \quad 0P, \boxed{1P}, 2P, \ldots\ldots$

  $h^0(0P) = 1, \quad h^0((2g-1)P) = 1 - g + (2g-1) = g.$

  $0P, \ 1P, \ 2P, \ \ldots\ldots, \ (2g-1)P \qquad g \geq 2$

  $h^0 = 1 \qquad\qquad h^0 = g$

  $\Rightarrow \ \forall g \geq 2, \ \exists n > 0$ s.t. nP is not basept free

- $\deg D \geq 2g \Rightarrow \mathcal{O}_D$ is generated by global secth ($|D|$ is basepoint free)

  Want: $h^0(\mathcal{O}_D) = h^0(\mathcal{O}_{D-p}) + 1 \quad \forall p \in X$

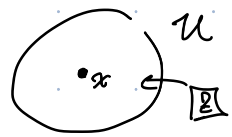
  $\deg \geq 2g-1 \Rightarrow h^1 = 0.$

  $h^0(\mathcal{O}_D) = 1 - g + \deg(D)$

  $h^0(\mathcal{O}_{D-p}) = 1 - g + \deg(D-p) = 1 - g + \deg(D) - 1$

Maps into $\mathbb{P}^N$: $f_0, \dots, f_N \in \mathcal{M}(X)^{\times}$.

Let $x \in X$, z : local coord. at x w/ $z(x) = 0$.



Let $k := \min_i \operatorname{ord}_x(f_i)$

Then $\left\{ g_i := \dfrac{f_i}{z^k} \right\}$ holo. in $\mathcal{U}$, and $g_i(x) \neq 0$ for some i.

Define $F(x) := [g_0(x), \dots, g_N(x)] \in \mathbb{CP}^N$.

Ex: $F : X \longrightarrow \mathbb{CP}^N$ is well-defined and holomorphic.

Def:
- F is an immersion if $dF_x : T_x X \longrightarrow T_{F(x)} \mathbb{P}^N$ injective $\forall x$.

  i.e. $\forall x \in X$, $\exists$ chart $\mathcal{U}_i \cong \mathbb{C}^N \subseteq \mathbb{CP}^N$

  s.t.
$$\begin{array}{ccccc} F: & X & \longrightarrow & \mathbb{CP}^N & \\ & \cup & & \cup & \\ x \in & F^{-1}\mathcal{U}_i & \longrightarrow & \mathcal{U}_i \cong \mathbb{C}^N & \exists j \text{ st. } dF_{ij}(x) \neq 0 \\ & x & \longmapsto & (F_{i1}(x), \dots, F_{iN}(x)) & \end{array}$$

- F is an embedding if F is an injective immersion

  (seperate points, seperate tangent vectors).

Thm X - cpt R.S. of genus g, D, st. $\boxed{\deg D \geq 2g+1}$

Let $f_0, \dots, f_N$ be a basis of $\boxed{H^0(X, \mathcal{O}_D)}$.

Then $F = [f_0, \dots, f_N] : X \longrightarrow \mathbb{CP}^N$ is an embedding

① Injectivity (separate points).

$x_1 \neq x_2$. " $F(x_1) = F(x_2)$ "

$\Leftrightarrow$ " $f_i(x_1) = \lambda f_i(x_2)$ $\forall i = 0, \dots, N$ for some $\lambda \in \mathbb{C}^\times$ "

$\Leftrightarrow$ " $f(x_1) = \lambda f(x_2)$ $\forall f \in H^0(X, \mathcal{O}_D)$ for some $\lambda \in \mathbb{C}^\times$ "

($f_0, \dots, f_N$ basis of $H^0(X, \mathcal{O}_D)$)

$X \to \mathbb{P}^N$ injective

$$\boxed{\begin{array}{ccc} H^0(X, \mathcal{O}_D) & \xrightarrow{ev_{x_1, x_2}} & \mathbb{C}^2 \\ F & \longmapsto & (F(x_1), F(x_2)) \end{array} \quad \text{surjective} \ \forall x_1 \neq x_2}$$

$$0 \to \mathcal{O}_{D - x_1 - x_2} \to \mathcal{O}_D \to \mathbb{C}_{x_1} \oplus \mathbb{C}_{x_2} \to 0$$

$$0 \to H^0(\mathcal{O}_{D - x_1 - x_2}) \to H^0(\mathcal{O}_D) \xrightarrow{ev_{x_1, x_2}} \mathbb{C}^2$$

$$\to \boxed{H^1(\mathcal{O}_{\underbrace{D - x_1 - x_2}})} \overset{?}{=} 0$$

$\deg \geq (2g+1) - 2 = 2g - 1$

② Immersion (separate tangent vector)

$x \in X$, WLOG $\longmapsto \left(\frac{f_1(x)}{f_0(x)}, \dots, \frac{f_N(x)}{f_0(x)} \right) \in U_0 \subseteq \mathbb{CP}^N$.

Want: $\exists k^{=1,\dots,N}$ s.t. $\left(\frac{f_k}{f_0}\right)'(x) \neq 0$.

$\Leftrightarrow$ $\boxed{f_k'(x) f_0(x) - f_k(x) f_0'(x) \neq 0.}$

Let $s := f_0(x) f_k - f_k(x) f_0 \in H^0(X, \mathcal{O}_D)$.

Then $s(x)=0$ $\Downarrow$ $s'(x)\neq 0$

To prove immersion at $x \longleftrightarrow \exists s \in H^0(X,\mathcal{O}_D)$ s.t. $s(x)=0$, $s'(x)\neq 0$

the first 2 coeff. of Laurent series of x

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{D-2x} \arrow[r] & \mathcal{O}_D \arrow[r] & \mathcal{O}_D \otimes \left(\mathcal{O}/I_x^2\right) \arrow[r] & 0 \end{tikzcd}$$

$\frac{a_{-n}}{z^n}+\frac{a_{-(n-1)}}{z^{n-1}}+\cdots \longmapsto (a_{-n}, a_{-(n-1)})$

I_x: ideal sheaf of holo. fcn. vanishing to order ≥ 2 at x.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{D-x} \arrow[r] & \mathcal{O}_D \arrow[r] & \mathcal{O}_D \otimes \mathbb{C}_x \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{-x} \arrow[r] & \mathcal{O} \arrow[r] & \mathbb{C}_x \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{-2x} \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}/I_x^2 \arrow[r] & 0 \end{tikzcd}$$

$\otimes \mathcal{O}_D$

$H^0(\mathcal{O}_D) \twoheadrightarrow H^0(\mathcal{O}_D \otimes \mathcal{O}/I_x^2)$ surjective

$H^1(\mathcal{O}_{D-2x}) = 0.$

$\deg \geq 2g-1$

$s = a_0 + a_1 z + a_2 z^2 + \cdots$

$(s(x), s'(x))$

$\quad \| \qquad \|$

$\quad a_0 \qquad a_1$

$a_0 = 0$

$s' = a_1 + 2a_2 z + \cdots$

"Canonical embedding" $g(X)\geq 2$, $\deg(K) = \boxed{2g-2} \geq 2$

Thm $g\geq 2$. K canonical divisor

- $|K|$ $\boxed{\text{basepoint free}}$, of dim. $g-1$.
- $\varphi_{|K|}: X \longrightarrow \mathbb{CP}^{g-1}$ is either an embedding or:

X is hyperelliptic (i.e. X admits a $\xrightarrow{2:1} \mathbb{P}^1$)

Rmk: ◦ $g=2. \to X$ hyperelliptic

- we'll sketch that for $g \geq 3$, most R.S. are not hyperelliptic

pf: • Basept free: $h^0(\mathcal{O}_K) = h^0(\mathcal{O}_{K-p}) + 1 \quad \forall p \in X$

- $h^0(\mathcal{O}_K) = h^1(\mathcal{O}) = g$
- $h^0(\mathcal{O}_{K-p}) - h^0(\mathcal{O}_p) = 1 - g + \deg(K-p) = \overset{\geq g-2-1}{g-2}$

RR

Need: $h^0(\mathcal{O}_p) = 1$, i.e. consists of only constant functions. $f \in H^0(X, \mathcal{O}_p)$

$f: X \longrightarrow \mathbb{P}^1$ deg 1 $\Rightarrow X \cong \mathbb{P}^1$ ✗

$\dim |K| = \dim \mathbb{P} H^0(X, \mathcal{O}_K) = g-1$

$0 \to \mathcal{O}_{K-2x} \to \mathcal{O}_K \to \mathcal{O}_K \otimes \mathcal{O}/\mathfrak{m}_x^2 \to 0$

$0 \to \mathcal{O}_{K-x_1-x_2} \to \mathcal{O}_K \to \mathbb{C}_{x_1} \oplus \mathbb{C}_{x_2} \to 0$

$0 \to H^0(\mathcal{O}_{K-x_1-x_2}) \to H^0(\mathcal{O}_K) \to \mathbb{C}^2 \to 0$

- When is $\varphi_{|K|}: X \longrightarrow \mathbb{CP}^{g-1}$ an embedding?

(i) Sep. pts: $\forall x_1 \neq x_2$, $H^0(\mathcal{O}_K) \xrightarrow{\text{ev.}} \mathbb{C}_{x_1} \oplus \mathbb{C}_{x_2}$ surjective.

$$\Longleftrightarrow \quad h^0(\mathcal{O}_{K-x_1-x_2}) = \overset{= g}{h^0(\mathcal{O}_K)} - 2.$$

(ii) Sep. tangent vect.: $\Longleftrightarrow \quad h^0(\mathcal{O}_{K-2x}) = h^0(\mathcal{O}_K) - 2.$

(i) ~~$h^0(\mathcal{O}_{K-x_1-x_2}) - h^0(\mathcal{O}_{x_1+x_2}) = 1 - g + \deg(K-x_1-x_2)$~~

$$\boxed{h^0(\mathcal{O}_{x_1+x_2})} - h^0(\mathcal{O}_{K-x_1-x_2}) = 1 - g + 2 = 3 - g.$$

So, "$h^0(\mathcal{O}_{K-x_1-x_2}) = h^0(\mathcal{O}_K) - 2$" $\Longleftrightarrow$ "$h^0(\mathcal{O}_{x_1+x_2}) = 1$"

① $h^0(\mathcal{O}_{x_1+x_2})=1 \quad \forall x_1\neq x_2 \quad \Rightarrow \quad X \xrightarrow{\varphi_{|K|}} \mathbb{P}^N$ embedding

② $h^0(\mathcal{O}_{x_1+x_2})>1$ for some $x_1\neq x_2$

i.e. $\exists$ non const mero. fun. with simple pole at x_1, x_2.

$\rightsquigarrow \quad f: X \longrightarrow \mathbb{P}^1$ holo. $\deg = 2$

$\Rightarrow$ X is hyperelliptic

(ii) Seperate ~~point~~ tangent vector $\Longleftrightarrow$ " $h^0(\mathcal{O}_{2x})=1 \quad \forall x\in X$ "

... proceed similarly ...

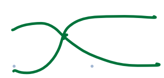
Rmk: Heuristic dim. count of $\mathcal{M}_g$ and $\mathcal{M}_{\text{hyperelliptic}, g}$

(Riemann)

$\mathcal{M}_g$: moduli space of cpt Rie. surf of genus g

$\mathcal{M}_{\text{hyperelliptic}, g}$: moduli space of hyperellipt. R.S. genus g.

① Hyperelliptic curve genus g: $X \xrightarrow{2:1} \mathbb{P}^1$



A generic hyperelliptic curve has exactly $2g+2$ branched pts

RH: $2g-2 = \underline{2\cdot(-2)} + \underbrace{\boxed{B}}_{2g+2}$

$\boxed{\mathrm{Aut}(\mathbb{P}^1)} \cong PGL_2(\mathbb{C}) \quad \dim_{\mathbb{C}}=3$

$\dim \mathcal{M}_{hyp} = (2g+2)-3 = 2g-1$

② $\dim \mathcal{M}_g = 3g-3$ $\qquad$ $d>>0$

Riemann: $\mathcal{H}_{d,g} \supseteq \{ f: X \to \mathbb{P}^1 \text{ of deg } d \}$

$\mathbb{P}^1 \ni \infty$

$\boxed{D} = f^{-1}\infty$ $\quad$ d pts

$H^0(X, \mathcal{O}_D)$ $\quad$ $d + h^0(\mathcal{O}_D)$

$1-g+d = 1-g+\deg D$

$$\begin{array}{c} \mathcal{H}_{d,g} \\ \downarrow \\ \mathcal{M}_g \ni X \end{array}$$

" $\dim \mathcal{H}_{d,g}$ " $-$ " dim of fibers of $\mathcal{H}_{d,g} \to \mathcal{M}_g$ "

$2g-2 = d(-2) + B$, $\quad B = 2g+2d-2$

$= 1-g+2d$

$= 3g-3$

$\Rightarrow$ $g \geq 3$ $\quad$ $3g-3 > 2g-1$ $\rightsquigarrow$ most elts of $\mathcal{M}_g$ are not hyperelliptic

Jacobian variety of a cpt. R.S. X $g(X) \geq 1$.

- **Algebraic def^n:**

$$\begin{array}{ccc} M_X^\times & \longrightarrow & \underline{\mathrm{Div}_0(f)} \\ f & \longmapsto & (f). \end{array}$$

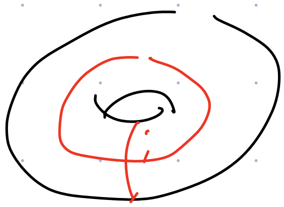
$$\mathrm{Pic}^0(X) := \mathrm{Div}_0(X) / \text{linear equivalence}$$

- **Analytic def^n:**

global holo. 1-form. (dim $= g$)

$\mathbb{Z}^{2g}.$

$$\begin{array}{ccc} H_1(X, \mathbb{Z}) & \xrightarrow{\iota} & H^0(X, \mathcal{O}^1)^* \\ [\gamma] & \longmapsto & \left(\omega \longmapsto \int_\gamma \omega\right) \end{array}$$



$$\mathrm{Jac}(X) := \frac{H^0(X, \mathcal{O}^1)^*}{\iota(H_1(X, \mathbb{Z}))}$$

Explicity, let $\omega_1, \dots, \omega_g$ be a basis of $H^0(X, \mathcal{O}^1)$

Let $\Lambda := \left\{ \left(\int_\gamma \omega_1, \dots, \int_\gamma \omega_g \right) \in \mathbb{C}^g \,\middle|\, \gamma \in \underline{H_1(X, \mathbb{Z})} \right\}$

Ex: $\Lambda \cong \mathbb{Z}^{2g} \subseteq \mathbb{C}^g$ is a lattice

i.e. The image of $\Lambda \subseteq \mathbb{C}^g$ is not contained in any real hyperplane

$\mathrm{Jac}(X) \cong \mathbb{C}^g / \Lambda$ is a complex g-dim$^{'l}$ torus.

Rmk. • Fix a basept $x_0 \in X$.

$$\begin{aligned} u: X &\longrightarrow \mathrm{Jac}(X) \\ p &\longmapsto \left(\int_{x_0}^{p} \omega_1, \dots, \int_{x_0}^{p} \omega_g \right) \end{aligned}$$

well-def. holo map

called the Abel-Jacobi map.

• Changing the basept $\longleftrightarrow$ translation in $\mathbb{C}^g/\Lambda$.