<u>Def:</u> X: R.S. The group of <u>divisors</u> $\mathrm{Div}(X)$. is the free abel. gp. generated by the points of X.

$$D = \sum_{P \in X} a_P P \quad \text{where } a_P \in \mathbb{Z}, \text{ and } a_P = 0 \text{ for all but finitely many pts.}$$

- A divisor is <u>effective</u> if $D \geq 0$, i.e. $a_P \geq 0 \ \forall P \in X$.
- Any divisor can be uniquely written as $D = D_1 - D_2$ where D_1, D_2 effective.
- Write $D \leq E$ if $E - D \geq 0$.
- <u>Degree</u> $\deg(D) := \sum a_P$. $\rightsquigarrow$ $\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}$ gp homom.
- $\mathrm{Div}_0(X)$: subgp of divisors with degree 0.

<u>Def:</u> X: R.S.

- For a mero. fcn $f \in \mathcal{M}(X) \setminus \{0\}$, <u>the divisor of f</u>, denoted by (f), is defined as $$(f) := \sum_{\substack{P:\text{ zero} \\ \text{or pole of } f}} \mathrm{ord}(f, P) \cdot P$$

  Divisors arise in this way are called <u>principal divisors</u>.

- Two divisors D_1, D_2 are called <u>equivalent</u> if $D_1 - D_2$ is principal.
- For a mero. 1-form $\omega \in \mathcal{M}^1(X) \setminus \{0\}$, one can similarly define its divisor $(\omega) \in \mathrm{Div}(X)$. Divisors arise in this way are called <u>canonical divisors.</u>
- <u>Note:</u> Any two canonical divisors are equivalent:
  ($\forall\, \omega_1, \omega_2$, $\exists\, f \in \mathcal{M}(X) \setminus \{0\}$ s.t. $\omega_1 = f \omega_2$. $\Rightarrow (\omega_1) - (\omega_2) = (f)$).

Def: X: cpt. R.S.

$$\deg: \mathrm{Div}(X) \longrightarrow \mathbb{Z}. \quad \text{gp homom.}$$
$$D = \sum a_p P \longmapsto \sum a_p.$$

- For principal divisor (f), $\deg(f) = 0$.

$\Rightarrow$ equivalent divisors have the same degree.

- For canonical divisor (ω), $\deg(\omega) = 2g-2$.

- $\mathcal{M}(X)^\times \longrightarrow \mathrm{Div}_0(X)$ is a gp homom.

  $f \longmapsto (f)$. $\qquad (fg) = (f) + (g)$.

  The cokernel of this homom is closely related to the Jacobian of X.
  (g-dim'l cpx. torus)

Def: $\mathcal{O}_D :=$ sheaf of mero. fcns s.t. $(f) + D \geq 0$.

(e.g. $\mathcal{O}_{np}(X) =$ mero. fcns on X w/ pole of order $\leq n$ at p.)

(in general, $\mathcal{O}_D$ keeps the pole order from being too big, and forces some zeros at pts where $a_p < 0$.)

- If $D \sim E$ equivalent, i.e. $D - E = (f)$, then $\mathcal{O}_D \cong \mathcal{O}_E$:
$$h \mapsto hf.$$

$$(hf) + E = (h) + (f) + E = (h) + D,$$

$$\Rightarrow \quad (h) + D \geq 0 \iff (hf) + E \geq 0.$$

- $\omega \in \mathcal{M}^1(X)^\times \rightsquigarrow$ a canonical divisor $(\omega) = K$

$$\mathcal{O}_{(\omega)} \cong \mathcal{O}^1 \leftarrow \text{sheaf of holo. 1-forms}$$
$$h \longmapsto h\omega.$$

$h\omega$ holo. $\iff (h\omega) = (h) + (\omega) \geq 0 \iff h \in \mathcal{O}_{(\omega)}$

Def: $\Omega^1_D :=$ sheaf of mero. 1-forms w/ $(\omega)+D \geq 0$.

Then $\Omega^1_D \cong \mathcal{O}_{K+D}$

Riemann-Roch problem: Compute/estimate $h^0(\mathcal{O}_D) = \dim H^0(X, \mathcal{O}_D) = \dim \mathcal{O}_D(X)$.

i.e. global mero. fcn w/ prescribed zeros/poles behavior.

e.g. X: cpt. R.S. $D \in \mathrm{Div}(X)$ with $\deg D < 0$. Then $h^0(\mathcal{O}_D) = 0$.

($f \in \mathcal{O}_D(X)$ is a global mero. fcn w/ $(f)+D \geq 0 \Rightarrow \deg(f) > 0$. $\cancel{\times}$)

Def: $x \in X$. The skyscraper sheaf $\mathbb{C}_x$ on X is:

$$\mathbb{C}_x(U) = \begin{cases} \mathbb{C} & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

- $H^0(X, \mathbb{C}_x) \cong \mathbb{C}$ and $H^1(X, \mathbb{C}_x) = 0$.

$H^0(X, \mathbb{C}_x) = \mathbb{C}_x(X)$

Suppose $(f_{ij}) \in Z^1(\mathcal{U}, \mathbb{C}_x)$. Choose a refinement $\mathcal{U} < \mathcal{V}$ s.t. x is contained in just one V_{i_0}. Then:

$$\begin{array}{ccccc} \prod \mathbb{C}_x(V_i) & \xrightarrow{\delta^0} & \prod \mathbb{C}_x(V_{ij}) & \xrightarrow{\delta^1} & \prod \mathbb{C}_x(V_{ijk}) \\ \| & & \| & & \| \\ \mathbb{C}_x(V_{i_0}) & & \mathbb{C}_x(V_{i_0} \cap V_{i_0}) & & \mathbb{C}_x(V_{i_0} \cap V_{i_0} \cap V_{i_0}) \\ & & \| & & \| \\ & & 0 & & 0 \end{array}$$

Similarly, we prove $H^p(X, \mathbb{C}_x) = 0 \ \forall p \geq 1$.

Lemma. $H^p(X, \mathcal{O}_D)$ is finite dim'l, and vanish $\forall p \geq 2$.

pf: • For $D=0$, $H^0(X,\mathcal{O}) \cong \mathbb{C}$, $H^1(X,\mathcal{O}) \cong \mathbb{C}^g$. For higher p:

$$0 \to \mathcal{O} \to \mathcal{E}^\infty_{\mathbb{C}} \xrightarrow{\bar\partial} \mathcal{A}^{0,1} \to 0 \quad \text{induces:}$$

$$0 \to H^0(X,\mathcal{O}) \to H^0(X, \mathcal{E}^\infty_{\mathbb{C}}) \to H^0(X, \mathcal{A}^{0,1})$$
$$\to H^1(X,\mathcal{O}) \to \cancel{0} \qquad \cancel{0}$$
$$\cancel{0} \qquad \cancel{0} \quad . \ \square$$

• For other D, use $0 \to \mathcal{O}_D \to \mathcal{O}_{D+x} \to \mathbb{C}_x \to 0$ (Ex:) short exact seq.

leading coeff of Laurent series at x.

$$\Rightarrow \quad 0 \to H^0(X,\mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+x}) \to \mathbb{C}$$
$$\to H^1(X,\mathcal{O}_D) \to H^1(X,\mathcal{O}_{D+x}) \to 0$$
$$\cong \qquad\qquad 0$$
$$\vdots$$
$$\cong \qquad\qquad \square$$

Lemma: $D_1 \leq D_2 \rightsquigarrow \mathcal{O}_{D_1} \hookrightarrow \mathcal{O}_{D_2} \rightsquigarrow \underline{H^1(X, \mathcal{O}_{D_1}) \twoheadrightarrow H^1(X,\mathcal{O}_{D_2})}$

($D_2 - D_1$ is effective, so we can obtain D_2 from D_1 by adding points, use $0 \to \mathcal{O}_{D_1} \to \mathcal{O}_{D_1+x} \to \mathbb{C}_x \to 0$ inductively.)

(Euler characteristic version)

Riemann-Roch: Let $\chi(\mathcal{O}_D) := h^0(X,\mathcal{O}_D) - h^1(X,\mathcal{O}_D)$.

Then: $\chi(\mathcal{O}_D) = \chi(\mathcal{O}) + \deg(D)$,

equivalently, $h^0(\mathcal{O}_D) - h^1(\mathcal{O}_D) = \deg D - g + 1$.

pf: Essentially follows from the above long exact seq.

$$\rightsquigarrow \chi(\mathcal{O}_{D+x}) = \chi(\mathcal{O}_D) + 1. \ \square$$

Easy consequences of R-R:

- Any cpt. R.S. X admits a nonconst. holo. map $X \xrightarrow{f} \mathbb{P}^1$ with $\deg(f) \leq g+1$.

  ($h^0(\mathcal{O}_D) \geq \deg D - g + 1$. $\Rightarrow$ If $\deg D > g$, then $h^0(\mathcal{O}_D) > 1 = \dim\left(\substack{\text{const.}\\ \text{map}}\right)$

  Such $f \in H^0(\mathcal{O}_D) \setminus \left\{\substack{\text{const.}\\ \text{fcn}}\right\}$ has at worst $g+1$ poles.

  $\deg D = g+1$

  Therefore the preimage of $\{\infty\}$ has at most $g+1$ points.)

Rmk: The bound $g+1$ is not optimal.

From Brill-Noether theory, one can improve it to $\lfloor \frac{g+3}{2} \rfloor$.

Mittag-Leffler problem for 1-forms:

Given mero. forms $\beta_i \in \mathcal{M}^1(U_i)$ s.t. $\beta_i - \beta_j \in \mathcal{O}^1(U_{ij})$ $\forall i,j$.

Does there exist $\omega \in \mathcal{M}^1(X)$ s.t. $\omega - \beta_i \in \mathcal{O}^1(U_i)$ $\forall i$. ?

(i.e. a global mero. 1-form with prescribed principal parts.)

- In terms of sheaves, the prescribed data is a global section $H^0(X, \mathcal{M}^1/\mathcal{O}^1)$, and we'd like to determine the image of $H^0(X, \mathcal{M}^1) \to H^0(X, \mathcal{M}^1/\mathcal{O}^1)$.
- From $0 \to \mathcal{O}^1 \to \mathcal{M}^1 \to \mathcal{M}^1/\mathcal{O}^1 \to 0$, the image is exactly the kernel

$$\ker\left(H^0(X, \mathcal{M}^1/\mathcal{O}^1) \xrightarrow{\delta} H^1(X, \mathcal{O}^1)\right).$$

- Recall that we have $0 \to \mathcal{O}^1 \to \mathcal{A}^{1,0} \xrightarrow{d} \mathcal{A}^2 \to 0$, and

$$H^1(X, \mathcal{O}^1) \cong \mathcal{A}^2(X) / \mathrm{Im}(\mathcal{A}^{1,0} \xrightarrow{d} \mathcal{A}^2) \cong H^{1,1}(X) \cong \mathbb{C}.$$

another connecting homom.

$$\omega \longmapsto \frac{1}{2\pi i} \int_X \omega.$$

- By Cauchy's thm, an obstruction for the existence of such global 1-form is that the sum of residues of (β_i) is zero. ↖ Claim. ⟺ "$\delta((\beta_i))=0$"

- Compute $H^0(X, \mathcal{M}^1_X/\mathcal{O}^1_X) \xrightarrow{\delta} H^1(X, \mathcal{O}^1_X)$:

$$H^0(X, \mathcal{M}^1_X/\mathcal{O}^1_X) \xrightarrow{\delta} H^1(X, \mathcal{O}^1_X) \;\text{1-dim}$$

$$H^1(X,\mathcal{O}^1_X) \cong \text{"}\delta\text{"}\; \mathcal{A}^2(X)/\ldots \cong \mathbb{C}$$

$$\begin{tikzcd}
C^0(\mathcal{U}, \mathcal{O}^1_X) \arrow[r] \arrow[d] & C^1(\mathcal{U}, \mathcal{O}^1_X) \arrow[d] \\
C^0(\mathcal{U}, \mathcal{M}^1_X) \arrow[r] \arrow[d] & C^1(\mathcal{U}, \mathcal{M}^1_X) \arrow[d] \\
C^0(\mathcal{U}, \mathcal{M}^1_X/\mathcal{O}^1_X) \arrow[r] & C^1(\mathcal{U}, \mathcal{M}^1_X/\mathcal{O}^1_X)
\end{tikzcd}$$

$(\beta_i - \beta_j) \in \mathcal{O}^1_X$, $(\beta_i) \mapsto (\beta_i - \beta_j)$, (β_i)

prescribed data of the M-C problem.

- Compute $H^1(X, \mathcal{O}^1) \xrightarrow{\cong} H^0(X, \mathcal{A}^2)/\mathrm{Im}(H^0(X, \mathcal{A}^{1,0}) \xrightarrow{\text{"}\bar\partial\text{"}} H^0(X, \mathcal{A}^2))$: $\cong H^{1,1}(X) \cong \mathbb{C}$

$$\begin{tikzcd}
C^0(\mathcal{U}, \mathcal{O}^1) \arrow[r] \arrow[d] & C^1(\mathcal{U}, \mathcal{O}^1) \arrow[d] \\
C^0(\mathcal{U}, \mathcal{A}^{1,0}) \arrow[r] \arrow[d] & C^1(\mathcal{U}, \mathcal{A}^{1,0}) \arrow[d] \\
C^0(\mathcal{U}, \mathcal{A}^2) \arrow[r] & C^1(\mathcal{U}, \mathcal{A}^2)
\end{tikzcd}$$

$(\alpha_{ij}) \mapsto 0$, (α_{ij}), $(\xi_i) \mapsto (\xi_i - \xi_j) \mapsto 0$, $\bar\partial\xi_i$, $\omega = (\omega_i) \mapsto 0$

$H^1(\mathcal{U}, \mathcal{A}^{1,0}) = 0$

- Initial data: (β_i) mero. w/ $(\beta_i - \beta_j)$ holo.

$$\begin{array}{ccccc}
H^0(X, \mathcal{M}^1/\mathcal{O}^1) & \longrightarrow & H^1(X, \mathcal{O}^1) & \cong & H^{1,1}(X) \\
(\beta_i) & \longmapsto & \alpha_{ij} = (\beta_i - \beta_j) & \longmapsto & \text{"}\boxed{\omega}\text{"}
\end{array}$$

  - Find (ξ_i) smooth $(1,0)$-forms s.t. $\alpha_{ij} = \xi_i - \xi_j$.

    (this can be done since $H^1(X, \mathcal{A}^{1,0}) = 0$.)

    Then: $\xi_i - \xi_j = \beta_i - \beta_j$ on U_{ij}.

    $\rightarrow$ $(\eta_i) = (\xi_i - \beta_i)$ is a global form.

- $\bar{\partial}\eta = \bar{\partial}\xi_i$ on U_i.

$\Rightarrow$ $\omega := \bar{\partial}\eta$ is the $(1,1)$-form that we've looking for.

- Summarize:

$$\begin{array}{ccccccc}
H^0(X, \mathcal{M}^1/\mathcal{O}^1) & \longrightarrow & H^1(X, \mathcal{O}^1) & \cong & H^{1,1}(X) & \xrightarrow{\ \cong\ } & \mathbb{C} \\
(\beta_i) & \longmapsto & (\beta_i - \beta_j) & \longmapsto & \bar{\partial}\eta & \longmapsto & \frac{1}{2\pi i}\int_X \bar{\partial}\eta \\
 & & & & & & \| \\
 & & & & & & \frac{1}{2\pi i}\int_X d(\underbrace{\xi_i - \beta_i}) \\
 & & & & & & \| \\
 & & & & & & -\sum \text{res. of } \beta_i
\end{array}$$

- $\exists$ smooth (ξ_i) s.t. $\xi_i - \xi_j = \beta_i - \beta_j$.
- Let $\eta := \xi_i - \beta_i$

mero. w/ poles of β_i

$\Rightarrow$ $\ker\left(H^0(X, \mathcal{M}^1/\mathcal{O}^1) \longrightarrow H^1(X, \mathcal{O}^1)\right) = \{(\beta_i) \mid \sum \text{res} = 0\}$

i.e. the sum of residues is the ONLY obstruction to the M-L problem for 1-forms.

e.g. • $\forall p_1 \neq p_2$ in X, $\exists$ mero. 1-form with simple poles of residues 1 and -1 at p_1, p_2. and no other poles.

- $\forall p \in X$, $n \geq 2$, $\exists$ mero. 1-form with pole of order n at p and no other poles.

<u>Mittag-Leffler problem for functions:</u> Given $\{p_1, \dots, p_n\} \subseteq X$ and Laurent tails $\quad f_i(z) = \frac{b_n}{z^n} + \dots + \frac{b_1}{z}$ at each p_i.

Does there exist a global mero. fcn. with the given principal parts?

- In terms of sheaves, we consider: $0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{M}/\mathcal{O} \to 0$

and:
$$\begin{array}{ccccc} H^0(X,\mathcal{M}) & \longrightarrow & H^0(X,\mathcal{M}/\mathcal{O}) & \longrightarrow & H^1(X,\mathcal{O}) \\ (f_i) & \longmapsto & (f_i) & \longmapsto & (f_i - f_j). \end{array}$$

- Recall that $0 \to \mathcal{O} \to \mathcal{C}^\infty \xrightarrow{\bar\partial} \mathcal{A}^{0,1} \to 0$ and

$$H^1(X,\mathcal{O}) \cong \mathcal{A}^{0,1}(X) / \mathrm{Im}(\mathcal{A}^0 \xrightarrow{\bar\partial} \mathcal{A}^{0,1}) =: H^{0,1}(X) \cong H^{1,0}(X)^*.$$

$$\begin{tikzcd} C^0(\mathcal{U},\mathcal{O}) \arrow[r] \arrow[d] & C^1(\mathcal{U},\mathcal{O}) \arrow[d] \\ C^0(\mathcal{U},\mathcal{C}^\infty) \arrow[r] \arrow[d] & C^1(\mathcal{U},\mathcal{C}^\infty) \arrow[d] \\ C^0(\mathcal{U},\mathcal{A}^{0,1}) \arrow[r] & C^1(\mathcal{U},\mathcal{A}^{0,1}) \end{tikzcd}$$

$(f_i - f_j)$; $(g_i) \mapsto (f_i - f_j)$; $(g_i) \mapsto (\bar\partial g_i)$

- (g_i) smooth fcn st. $\underset{\text{sm.}}{g_i - g_j} = \underset{\text{mero.}}{f_i - f_j}$.
- $(h_i := g_i - f_i)$ globally defined $\rightsquigarrow h$
- $\bar\partial h = \bar\partial g_i$ on U_i
- So, $H^1(X,\mathcal{O}) \xrightarrow{\cong} \mathcal{A}^{0,1}(X)/\mathrm{Im}\,\bar\partial$

  $(f_i - f_j) \longmapsto [\bar\partial h]$.

- M-L for (f_i) has sol$^{\underline{n}}$ $\iff$ $(f_i - f_j) = 0$ in $H^1(X,\mathcal{O})$

$\iff$ $[\bar\partial h] = 0$ in $\mathcal{A}^{0,1}(X)/\mathrm{Im}\,\bar\partial \cong H^{0,1}(X) \cong H^{1,0}(X)^*$

$\iff$ $\forall\, \omega \in \mathcal{O}^1(X) = H^{1,0}(X)$, $\displaystyle\int_X \omega \wedge \bar\partial h = 0.$

$h = \underbrace{g_i}_{\text{sm.}} - \underbrace{f_i}_{\text{mero.}}$

$\iff$ $\sum \mathrm{Res}_{p_i}(f_i\omega) = 0 \quad \forall\, \omega \in \mathcal{O}^1(X)$.

<u>Serre Duality</u>: There is a non-degenerate pairing:

$$H^0(X, \mathcal{O}^1_{-D}) \times H^1(X, \mathcal{O}_D) \longrightarrow H^1(X, \mathcal{O}^1) \cong \mathbb{C}$$

- There is: $\mathcal{O}^1_{-D} \times \mathcal{O}_D \longrightarrow \mathcal{O}^1$

  $(\omega, f) \longmapsto f\omega$

  which induces $H^0(X, \mathcal{O}^1_{-D}) \times H^1(X, \mathcal{O}_D) \longrightarrow H^1(X, \mathcal{O}^1) \cong \mathbb{C}$

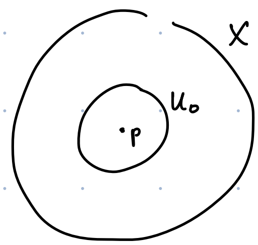
<u>Claim</u>: $H^0(X, \mathcal{O}^1_{-D}) \longrightarrow H^1(X, \mathcal{O}_D)^*$ is injective.

pf: For $\omega \in H^0(X, \mathcal{O}^1_{-D}) \setminus \{0\}$, i.e. $(\omega) - D \geq 0$.

<u>want</u>: Find $\xi \in H^1(X, \mathcal{O}_D)$ s.t. $\langle \omega, \xi \rangle \neq 0$.

- choose a pt $p \in X$ with $p \in U_0$ outside the supp. of D and (ω).

  choose a local coord. z s.t. $\omega = dz$.

- 

  Set $f_0 = \frac{1}{z}$ on U_0,

  $f_1 = 0$ on $U_1 := X \setminus \{p\}$

  $$H^0(\mathcal{U}, \mathcal{M}/\mathcal{O}_D) \xrightarrow{\delta} H^1(\mathcal{U}, \mathcal{O}_D).$$

  $$(f_0, f_1) \longmapsto \underbrace{f_0 - f_1}_{\text{holo. in } U_0 \cap U_1}$$

- By previous computations,

  $$\langle \omega, \xi \rangle = \operatorname{Res}\big(\underbrace{(f_0 - f_1)}_{\text{only nonzero in } U_0}\omega\big) = 1 \neq 0.$$

$\Longrightarrow h^0(\mathcal{O}_{K-D}) - h^1(\mathcal{O}_D) \leq 0.$

Similarly, replace D by $K-D$, one has $h^0(\mathcal{O}_D) - h^1(\mathcal{O}_{K-D}) \leq 0$.

$\Rightarrow \chi(\mathcal{O}_D) + \chi(\mathcal{O}_{K-D}) \leq 0$.

- $\chi(\mathcal{O}_D) = \chi(\mathcal{O}) + \deg(D) = 1-g+\deg(D)$

  $\chi(\mathcal{O}_{K-D}) = \chi(\mathcal{O}) + \deg(K-D) = \chi(\mathcal{O}) + 2g-2 - \deg(D). = -(1-g) - \deg(D)$

$\Rightarrow$ The equality must hold. $\square$

$\Rightarrow$ Serre duality

$\Rightarrow$ RR: $h^0(D) - h^0(K-D) = 1-g+\deg(D)$