* HW3 due Next week

Recap: • X, $\mathcal{U} = \{U_i\}_{i\in I}$ open cover, $\mathcal{F}$ sheaf on X

Čech cochain complex:

$$0 \to C^0(\mathcal{U},\mathcal{F}) \xrightarrow{\delta^0} C^1(\mathcal{U},\mathcal{F}) \xrightarrow{\delta^1} C^2(\mathcal{U},\mathcal{F}) \to \cdots$$

$$C^0(\mathcal{U},\mathcal{F}) = \prod \mathcal{F}(U_i), \quad C^1(\mathcal{U},\mathcal{F}) = \prod \mathcal{F}(U_{i_0}\cap U_{i_1}), \quad C^2(\mathcal{U},\mathcal{F}) = \prod \mathcal{F}(U_{i_0}\cap U_{i_1}\cap U_{i_2})$$

$$(f_{i_0 i_1}) \mapsto (g_{i_0 i_1 i_2}), \quad g_{i_0 i_1 i_2} = f_{i_1 i_2} - f_{i_0 i_2} + f_{i_0 i_1}$$

$\rightsquigarrow$ $$H^p(\mathcal{U},\mathcal{F}) = \frac{\ker \delta^p \ \text{cocycle}}{\operatorname{Im} \delta^{p-1} \ \text{coboundary}}$$

$\rightsquigarrow$ $$H^p(X,\mathcal{F}) := \varinjlim_{\mathcal{U}} H^p(\mathcal{U},\mathcal{F})$$

Fact: • $H^0(X,\mathcal{F}) = \mathcal{F}(X) = H^0(\mathcal{U},\mathcal{F})$

• $\mathcal{U} < \mathcal{V}$ refinement, $H^1(\mathcal{U},\mathcal{F}) \longrightarrow H^1(\mathcal{V},\mathcal{F})$ is injective

→ • (Leray) Suppose $H^1(U_i,\mathcal{F}) = 0$ $\forall i\in I$,

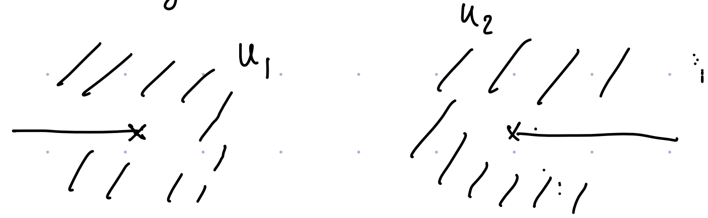
then $H^1(X,\mathcal{F}) \cong H^1(\mathcal{U},\mathcal{F})$

e.g. • $\mathcal{F}$ sheaf of modules over sheaf of $\mathcal{C}^\infty$-fcns on X ← partition of unity

$\Rightarrow$ $H^p(X,\mathcal{F}) = 0$ $\forall p \geq 1$.

• X : R.S. $\pi_1(X) = 0$, $H^1(X,\mathbb{C}) = H^1(X,\mathbb{Z}) = 0$.

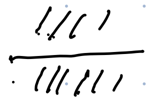
e.g. $\boxed{H^1(\mathbb{C}^\times, \underline{\mathbb{Z}})}$ Cover $\mathbb{C}^\times$ by:



This is a Leray cover $\Rightarrow$ $H^1(\mathbb{C}^\times, \underline{\mathbb{Z}}) \cong H^1(\mathcal{U}, \underline{\mathbb{Z}})$

$$\mathcal{U} = \{U_1, U_2\}$$

$$\begin{array}{ccccccc}
0 \longrightarrow & C^0(\mathcal{U}, \underline{\mathbb{Z}}) & \longrightarrow & C^1(\mathcal{U}, \underline{\mathbb{Z}}) & \longrightarrow & C^2(\mathcal{U}, \underline{\mathbb{Z}}) \\
& \| & & \| & & \| \\
& \underline{\mathbb{Z}}(U_1) \times \underline{\mathbb{Z}}(U_2) & & \underline{\mathbb{Z}}(\underbrace{U_1 \cap U_2}) & & 0 \\
& \wr\| & & & & \\
& \mathbb{Z} \times \mathbb{Z} & & & &
\end{array}$$



$$\wr\| \qquad \underline{\mathbb{Z} \times \mathbb{Z}}$$

$$(\underline{a_1}, \underline{a_2}) \longmapsto (\underline{a_2 - a_1, a_2 - a_1})$$

$$\Rightarrow \quad H^1(\mathcal{U}, \underline{\mathbb{Z}}) \cong \mathbb{Z}$$

$$\| $$

$$H^1(\mathbb{C}^\times, \underline{\mathbb{Z}})$$

e.g. $X \underset{\text{open}}{\subseteq} \mathbb{C}$ $\mathcal{O} \leftarrow$ sheaf of holo fcns on X

$$H^1(X,\mathcal{O}) = 0 \iff \boxed{H^1(\mathcal{U},\mathcal{O}) = 0 \ \forall \text{ open cover } \mathcal{U}}$$

pf:

$$0 \to C^0(\mathcal{U},\mathcal{O}) \to C^1(\mathcal{U},\mathcal{O}) \to C^2(\mathcal{U},\mathcal{O})$$

$$\text{"}(g_i)\text{"} \longmapsto \boxed{(f_{ij}) \longmapsto 0}$$

$$\boxed{\mathcal{O}} \subseteq \boxed{\mathcal{C}^\infty_{\mathbb{C}}}$$

- Since $H^1(\mathcal{U},\mathcal{C}^\infty_{\mathbb{C}}) = 0$

  $\Rightarrow \exists (g_i) \in C^0(\mathcal{U},\mathcal{C}^\infty_{\mathbb{C}})$ s.t. $f_{ij} = g_i - g_j$ on U_{ij}.

- $\bar\partial f_{ij} = 0 \Rightarrow \bar\partial g_i = \bar\partial g_j$ on U_{ij}

  $\Rightarrow \exists h \in \mathcal{C}^\infty(X)_{\mathbb{C}}$ s.t. $h|_{U_i} = \bar\partial g_i$

Fact (Dolbeault lemma) $X \underset{\text{open}}{\subseteq} \mathbb{C}$, $f \in \mathcal{C}^\infty(X)_{\mathbb{C}}$

$\exists u \in \mathcal{C}^\infty(X)_{\mathbb{C}}$ s.t. $\bar\partial u = f$.

(e.g. f has cpt supp. then a solution u is:

$$u(z) := \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{f(w)}{w-z}\, dw \wedge d\bar w.\ )$$

- $\exists g \in \mathcal{C}^\infty(X)_{\mathbb{C}}$ s.t. $\bar\partial g = h = \bar\partial g_i$ on U_i

• Let $\phi_i := g_i - g$ $\Rightarrow$ $\bar{\partial}\phi_i = \bar{\partial} g_i - \bar{\partial} g = 0$ on U_i

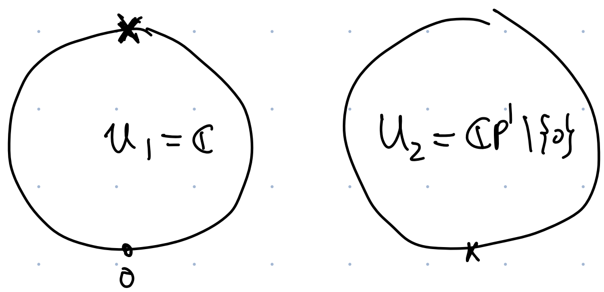
$\rightsquigarrow (\phi_i) \in C^0(\mathcal{U}, \mathcal{O})$

• $$\begin{array}{ccc} (\phi_i) & \xrightarrow{\ \delta_0\ } & (f_{ij}) \\ \cap & & \cap \\ C^0(\mathcal{U},\mathcal{O}) & & C^1(\mathcal{U},\mathcal{O}) \end{array}$$

$$f_{ij} = g_i - g_j \text{ on } U_{ij} = (g_i - g) - (g_j - g) = \phi_i - \phi_j \qquad \square$$

Rmk: In fact, $H^1(X, \mathcal{O}) = 0$ for X non-compact.

e.g. $H^1(\mathbb{CP}^1, \mathcal{O}) = 0$



$\mathcal{U} = \{U_1, U_2\}$ is a Leray cover of $\mathcal{O}$ on $\mathbb{CP}^1$

$\Downarrow$

$H^1(\mathbb{CP}^1, \mathcal{O}) \cong H^1(\mathcal{U}, \mathcal{O})$

$$\begin{array}{ccccccccc} 0 \to & C^0(\mathcal{U},\mathcal{O}) & \to & C^1(\mathcal{U},\mathcal{O}) & \to & C^2(\mathcal{U},\mathcal{O}) & \to \cdots \\ & \| & & \| & & \| & \\ & \mathcal{O}(U_1) \times \mathcal{O}(U_2) & & \mathcal{O}(U_1 \cap U_2) & & 0 & \\ & & & \| & & & \\ & & & \mathbb{C}^\times & & & \end{array}$$

$$\left(\sum_{n=0}^{\infty} a_n z^n ,\ -\sum_{n=-\infty}^{-1} a_n z^n \right) \longmapsto f = \sum_{n=-\infty}^{\infty} a_n z^n$$

Def $\mathcal{F}_1, \mathcal{F}_2$ sheaves on X

A sheaf homomorphism $\alpha: \mathcal{F}_1 \to \mathcal{F}_2$ is a collection of gp homom.

$\{\alpha_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)\}$ compatible with restriction maps:

$\forall$ $V \subseteq U$,

$$\begin{tikzcd} \mathcal{F}_1(U) \arrow[r, "\alpha_U"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}_2(U) \arrow[d, "\delta"] \\ \mathcal{F}_1(V) \arrow[r, "\alpha_V"'] & \mathcal{F}_2(V) \end{tikzcd}$$

e.g.
- $d: \mathcal{A}^i \longrightarrow \mathcal{A}^{i+1}$ derivative on sheaf of C^∞-forms.
- $\mathcal{O} \hookrightarrow \mathcal{C}^\infty_{\mathbb{C}}$, $\mathbb{C} \hookrightarrow \mathcal{C}^\infty_{\mathbb{C}}$
- $\exp: \mathcal{O} \longrightarrow \mathcal{O}^\times$ ← multiplicative sheaf of holo. fcns w/ value in $\mathbb{C}^\times$

where $\exp_U : \mathcal{O}(U) \longrightarrow \mathcal{O}^\times(U)$

$$f \longmapsto \exp(2\pi i f)$$

Def $\alpha: \mathcal{F}_1 \to \mathcal{F}_2$ sheaf homom.

Define $\ker(\alpha)$ (also a sheaf on X). by:

$$\ker(\alpha)(U) := \ker\left(\alpha_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)\right)$$

Ex: Verify $\ker(\alpha)$ is a sheaf.

e.g. • $\ker\left(\mathcal{O} \xrightarrow{\exp} \mathcal{O}^{\times}\right) \cong \underline{\mathbb{Z}}$

• $\ker\left(\mathcal{C}^{\infty}_{\mathbb{C}} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}\right) \cong \mathcal{O}$

Def: $\alpha: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ sheaf homom.

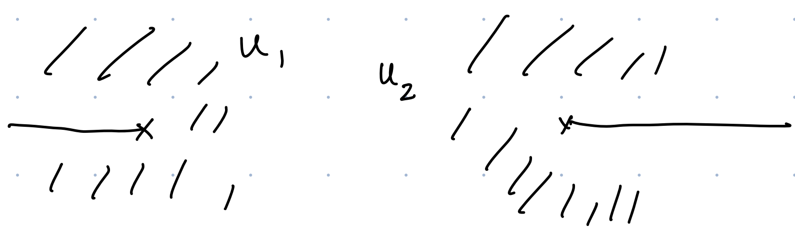
Define $\operatorname{Im}(\alpha)$ (presheaf on X) by:

$$\operatorname{Im}(\alpha)(U) := \operatorname{Im}\left(\alpha_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)\right).$$

However, $\operatorname{Im}(\alpha)$ is not a sheaf in general.

(usually, we refer $\operatorname{Im}(\alpha)$ as the "sheafification" of this presheaf.)

e.g. $X = \mathbb{C}^{\times}$, $\exp: \mathcal{O} \longrightarrow \mathcal{O}^{\times}$



$f_1 \in \mathcal{O}^{\times}(U_1)$ $\qquad$ $f_2 \in \mathcal{O}^{\times}(U_2)$

$f_1(z) = z$ $\qquad$ $f_2(z) = z$

• Since U_1, U_2 are both simply connected, so "log" can be defined on U_1, U_2, $\Rightarrow f_1 \in (\operatorname{Im}\alpha)(U_1)$, $f_2 \in (\operatorname{Im}\alpha)(U_2)$.

• $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$

- If $\mathrm{Im}\,\alpha$ were a sheaf on $X = \mathbb{C}^\times$, then
  $\exists\ f \in (\mathrm{Im}\,\alpha)(X)$ s.t. $\underbrace{f|_{U_i} = f_i}_{}$ $\rightsquigarrow f(z) = z$ on $\mathbb{C}^\times$

But log cannot be defined on $\mathbb{C}^\times$. so such f doesn't exist,

<u>Rmk</u>: $\underset{\text{presheaf}}{\mathcal{F}} \xrightarrow{\text{"sheafification"}} \underset{\text{sheaf.}}{\mathcal{F}^+}$

- If a collection of local sections agree on overlaps, sheafification forces their "glued" section to exist.
- In practice, $s \in \mathcal{F}^+(U)$ <u>locally</u> looks like section of $\mathcal{F}$.
- In the above example, $\underline{\mathrm{Im}(\exp)^+ = \boxed{\mathcal{O}^\times}}$

$\exists\ g$ hol.
$\exp(g) = f$
f hol.

x simply conn

$\rightsquigarrow$ exact seq. of sheaves:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \longrightarrow 0$$

<u>Def</u> A seq. of sheaves $\mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3$ is <u>exact</u>.
if $\mathcal{F}_{1,x} \xrightarrow{\alpha_x} \mathcal{F}_{2,x} \xrightarrow{\beta_x} \mathcal{F}_{3x}$ is exact $\forall x \in X$

<u>Ex</u>: $\mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2$ is injective, i.e. $\mathcal{F}_{1,x} \xrightarrow{\alpha_x} \mathcal{F}_{2,x}$ inj. $\forall x \in X$
then $\alpha_U : \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U)$ is injective

Rmk: exp: $\mathcal{O} \longrightarrow \mathcal{O}^\times$ surjective. ($\exp_x : \mathcal{O}_x \longrightarrow \mathcal{O}_x^\times$ surjective)

But $\exp_{\mathbb{C}^\times} : \mathcal{O}(\mathbb{C}^\times) \longrightarrow \mathcal{O}^\times(\mathbb{C}^\times)$ is <u>not</u> surjective

Ex: $0 \to \mathcal{F}_1 \to \mathcal{G}_2 \to \mathcal{F}_3$ exact

Then $\forall U \subseteq X$, $0 \to \mathcal{F}_1(U) \to \mathcal{G}_2(U) \to \mathcal{F}_3(U)$ exact.

(i.e., taking global section is a left exact functor)

e.g. X - R.S. the following are exact sequences of sheaves on X:

- $0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 0$

- $0 \to \mathcal{O} \to \mathcal{C}^\infty_{\mathbb{C}} \xrightarrow{\bar\partial} \mathcal{A}^{0,1} \longrightarrow 0$ Dolbeault lemma

(in general, $0 \to \mathcal{C}^\infty_{\mathbb{C}} \xrightarrow{\bar\partial} \mathcal{A}^{0,1} \xrightarrow{\bar\partial} \mathcal{A}^{0,2} \xrightarrow{\bar\partial} \cdots$ is exact.)

- $0 \to \underline{\mathbb{C}} \to \mathcal{C}^\infty_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^1_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^2_{\mathbb{C}} \longrightarrow 0$

(Poincaré lemma)

- $0 \to \underline{\mathbb{C}} \to \mathcal{O} \xrightarrow{d} \mathcal{O}^1 \longrightarrow 0$

sheaf of holo 1-forms

- $0 \to \mathcal{O}^1 \to \mathcal{A}^{1,0} \xrightarrow{d} \mathcal{A}^2_{\mathbb{C}} \longrightarrow 0$

Rmk: $\underline{\alpha : \mathcal{F}_1 \to \mathcal{F}_2 \text{ sheaf homom.}}$

- $\mathcal{U}$ open cover of X.

$$\begin{tikzcd}
\to \underline{C^p(\mathcal{U}, \mathcal{F}_1)} = \prod \mathcal{F}_1(U_{i_0 \cdots i_p}) \arrow[r] \arrow[d, "\alpha^p"] \arrow[dr, phantom, "\circlearrowright"] & C^{p+1}(\mathcal{U}, \mathcal{F}_1) \arrow[r] \arrow[d, "\alpha^{p+1}"] & \cdots \\
\to \underline{C^p(\mathcal{U}, \mathcal{F}_2)} = \prod \mathcal{F}_2(U_{i_0 \cdots i_p}) \arrow[r] & C^{p+1}(\mathcal{U}, \mathcal{F}_1) \arrow[r] & \cdots
\end{tikzcd}$$

cocycle $\to$ cocycle

coboundary $\to$ coboundary

$$\rightsquigarrow \quad H^p(\mathcal{U}, \mathcal{F}_1) \longrightarrow H^p(\mathcal{U}, \mathcal{F}_2)$$

$$\rightsquigarrow \quad H^p(X, \mathcal{F}_1) \longrightarrow H^p(X, \mathcal{F}_2).$$

Thm A short exact seq. of sheaves $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ induces a long exact seq.

$$\begin{aligned}
0 \to H^0(X, \mathcal{A}) &\longrightarrow H^0(X, \mathcal{B}) \longrightarrow H^0(X, \mathcal{C}) \\
\to H^1(X, \mathcal{A}) &\longrightarrow H^1(X, \mathcal{B}) \longrightarrow H^1(X, \mathcal{C}) \\
\to H^2(X, \mathcal{A}) &\longrightarrow \cdots
\end{aligned}$$

Let's define $\underline{\text{connecting homomorphism}}$ $\boxed{\underbrace{H^0(X, \mathcal{C})}_{\ni f} \longrightarrow H^1(X, \mathcal{A})}$

• Since $\mathcal{B} \to \mathcal{C}$ surjective, $\exists$ open cover $\mathcal{U}$ of X and $(g_i) \in C^0(\mathcal{U}, \mathcal{B})$ s.t. $\beta(g_i) = f|_{U_i}$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}^0(\mathcal{U}, \mathcal{A}) \arrow[r] \arrow[d] & \mathcal{C}^1(\mathcal{U}, \mathcal{A}) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{C}^0(\mathcal{U}, \mathcal{B}) \arrow[r] \arrow[d] & \mathcal{C}^1(\mathcal{U}, \mathcal{B}) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{C}^0(\mathcal{U}, \mathcal{C}) \arrow[r] & \mathcal{C}^1(\mathcal{U}, \mathcal{C}) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
 & (h_{ij}) \arrow[r, mapsto] \arrow[d, mapsto] & \boxed{0} \arrow[d, mapsto] \\
(g_i) \arrow[r, mapsto] \arrow[d, mapsto] & (g_i - g_j) \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
f \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$

$$\begin{array}{l}
0 \\
\downarrow \\
\mathcal{C}^1(\mathcal{U}, \mathcal{A}) = \prod \mathcal{A}(U_{i_0 i_1}) \\
\downarrow \\
\mathcal{C}^1(\mathcal{U}, \mathcal{B}) = \prod \mathcal{B}(U_{i_0 i_1}) \\
\downarrow \\
\mathcal{C}^1(\mathcal{U}, \mathcal{C}) = \prod \mathcal{C}(U_{i_0 i_1})
\end{array}$$

$0 \to \mathcal{A}(U_{i_0 i_1}) \to \mathcal{B}(U_{i_0 i_1}) \to \mathcal{C}(U_{i_0 i_1})$

e.g. X. R.S. (cpt.)

- $0 \to \mathcal{O} \to \mathcal{C}^\infty_\mathbb{C} \xrightarrow{\bar\partial} \mathcal{A}^{0,1} \to 0$

$\rightsquigarrow \; 0 \to H^0(X,\mathcal{O}) \to H^0(X,\mathcal{C}^\infty_\mathbb{C}) \to H^0(X,\mathcal{A}^{0,1})$

$\quad \to H^1(X,\mathcal{O}) \to \underbrace{H^1(X,\mathcal{C}^\infty_\mathbb{C})}_{=0}$

$$\Longrightarrow \; H^1(X,\mathcal{O}) \cong \frac{\mathcal{A}^{0,1}(X)}{\bar\partial\, \mathcal{C}^\infty_\mathbb{C}(X)} =: H^{0,1}(X)$$

$$\begin{tikzcd}
 & A^{1,0}(X) \arrow[dr, "\bar\partial"] & \\
\mathcal{C}^\infty_\mathbb{C}(X) = A^0(X) \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"'] & & A^2(X)_\mathbb{C} \\
 & A^{0,1}(X) \arrow[ur, "\partial"'] &
\end{tikzcd}$$

- $0 \to \mathcal{O}^1 \to \mathcal{A}^{1,0} \xrightarrow{d=\bar\partial} \mathcal{A}^2_\mathbb{C} \to 0$

$\rightsquigarrow \; 0 \to H^0(X,\mathcal{O}^1) \to H^0(X,\mathcal{A}^{1,0}) \to H^0(X,\mathcal{A}^2_\mathbb{C})$

$\quad \to \boxed{H^1(X,\mathcal{O}^1)} \to H^1(X,\mathcal{A}^{1,0}) = 0 \to H^1(X,\mathcal{A}^2_\mathbb{C})$

$\quad \to H^2(X,\mathcal{O}^1) \to \cdots$

sheaf of mod./ sheaf of $\mathcal{C}^\infty$-fcn

$g(z) \cdot f(z)\,dz$

$$\Rightarrow \quad H^1(X, \mathcal{O}^1) \cong \frac{\mathcal{A}^2(X)_{\mathbb{C}}}{\bar\partial \mathcal{A}^{1,0}(X)} =: H^{1,1}(X).$$

$$\bullet \quad 0 \to \underline{\mathbb{C}} \longrightarrow \mathcal{C}^\infty_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^1_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^2_{\mathbb{C}} \longrightarrow 0$$

$$\rightsquigarrow \quad 0 \to \underline{\mathbb{C}} \to \mathcal{C}^\infty_{\mathbb{C}} \xrightarrow{d} \ker(\mathcal{A}^1_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^2_{\mathbb{C}}) \longrightarrow 0$$

$$\rightsquigarrow 0 \to H^0(X, \underline{\mathbb{C}}) \to H^0(X, \mathcal{C}^\infty_{\mathbb{C}}) \longrightarrow H^0(X, \ker(\mathcal{A}^1_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^2_{\mathbb{C}}))$$

$$\to H^1(X, \underline{\mathbb{C}}) \to \underset{\substack{\| \\ 0}}{H^1(X, \mathcal{C}^\infty_{\mathbb{C}})} \to \cdots$$

$$\Rightarrow \quad H^1(X, \mathbb{C}) \cong \frac{\ker(\mathcal{A}^1(X)_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^2(X)_{\mathbb{C}})}{\operatorname{Im}(\mathcal{A}^0(X)_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^1(X)_{\mathbb{C}})} \cong H^1_{dR}(X, \mathbb{C})$$

Rmk: $\bullet$ $H^1(X, \mathcal{O}) \cong H^{0,1}(X) \overset{\mathcal{S}_X}{\cong} H^{1,0}(X)^* \cong H^0(X, \mathcal{O}^1)^*$

$\bullet$ $H^1(X, \mathcal{O}^1) \cong H^{1,1}(X) \cong H^{0,0}(X)^* \cong H^0(X, \mathcal{O})^*$

These are special case of <u>Serre duality</u> (in the case of cpt. R.S)

Denote $\omega_X = \mathcal{O}^1$.

$$H^0(X, \mathcal{L}) \xleftrightarrow{\text{dual}} H^1(X, \omega_X \otimes \mathcal{L}^\vee) \quad \forall \text{ line bundle } \mathcal{L}$$

$$\underset{\mathcal{O}_D}{H^0(X, \mathcal{O}(D))} \xleftrightarrow{\text{dual}} H^1(X, \omega_X(-D)) \quad \forall \text{ divisor } D$$

<u>Def</u>: X : R.S. The group of <u>divisors</u> $\mathrm{Div}(X)$ is the free abelian group generated by the points of X.

$$D = \sum_{P \in X} a_P \cdot P \quad \text{where } a_P \in \mathbb{Z}, \text{ and}$$

$a_P = 0$ for all but finitely many P

- A divisor D is <u>effective</u> if $a_P \geq 0 \;\forall P$. ($\underline{D \geq 0}$)
- Any divisor D can be written as $D = D_1 - D_2$ where D_1, D_2 are effective
- Write "$D \leq E$" if $E - D$ effective ($E - D \geq 0$)
- <u>Degree</u>: $\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}$ gp homom.

$$D = \sum a_P \cdot P \longmapsto \sum a_P$$

- $\mathrm{Div}_0(X) =$ subgp of divisors of degree 0.

<u>Def</u>. • $f \in \mathcal{M}(X) \setminus \{0\} \rightsquigarrow$ the <u>divisor of f</u>, denoted by (f)

$$(f) := \sum_{\substack{P:\ \text{zero or} \\ \text{pole of } f}} \mathrm{ord}(f, P) \cdot P$$

($\deg(f) = 0$ for X - cpt. R.S.)

Such divisors are called <u>prinicipal divisors</u>.

- We say D_1, D_2 are <u>linearly equivalent</u> if $D_1 - D_2$ is principal.

- $\omega \in \mathcal{M}^1(X) \setminus \{0\}$, the divisor of ω, denoted (ω)

$$(\omega) := \sum_{\substack{P:\ \text{zero/pole} \\ \text{of } \omega}} \operatorname{ord}(\omega, P) \cdot P$$

($\deg(\omega) = 2g-2$ for X cpt RS. of genus g)

Such divisors are called canonical divisors

Rmk: Any two canonical divisor is linearly equivalent.

($\omega_1, \omega_2 \in \mathcal{M}^1(X) \setminus \{0\}$. $\exists f \in \mathcal{M}(X) \setminus \{0\}$ s.t. $\omega_1 = f\omega_2$

$\Rightarrow \quad (\omega_1) = (f) + (\omega_2)$)