§ Elliptic functions, Modular functions / forms.

$\uparrow$ (meromorphic) functions on $\mathbb{C}/\Lambda$

$\uparrow$ Functions / Forms on "moduli space of complex torus" $\cong$ $\mathbb{H}/SL(2,\mathbb{Z})$

Some Fun Applications:

- $\exp(\pi\sqrt{163}) \approx \boxed{\quad \mathbb{Z} \quad}.\underline{999\cdots\cdots 9}25\cdots\cdots$

- $\sigma_3(n) := \sum_{\substack{d \mid n \\ 1 \le d \le n}} d^3, \quad \sigma_7(n) = \sum_{\substack{d \mid n \\ 1 \le d \le n}} d^7$

n	1	2	3	$\cdots$
$\sigma_3(n)$	1	9	28	$\cdots$
$\sigma_7(n)$	1	129	2188	$\cdots$

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{\substack{p+q=n \\ 1 \le p,q < n}} \sigma_3(p)\,\sigma_3(q).$$

- Sum of 4 squares

$$r_4(n) := \#\{(x_1,x_2,x_3,x_4) \in \mathbb{Z}^4 \mid n = x_1^2 + \cdots + x_4^2\}$$

$$= 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d$$

- $\mathbb{C}/\Lambda$: complex torus (cpt. R.S. $g=1$)

- Fact: The only holomorphic functions on $\mathbb{C}/\Lambda$ are constant functions.
  (i.e. holo. elliptic functions)
  $f(z+\lambda) = f(z)\ \forall z \in \mathbb{C}, \lambda \in \Lambda$.

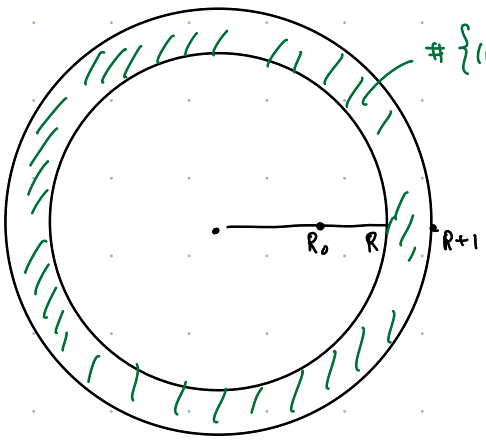
pf:  ω_2, ω_1 $\quad \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ $\quad$ Liouville thm $\Rightarrow$ const. fcn.

e.g. g - mero. fcn. on $\mathbb{C}$. (e.g. $g=\frac{1}{P(z)}$)

Consider $f(z) := \sum_{m,n\in\mathbb{Z}} g(\underline{z+m\omega_1+n\omega_2})$

When does $f(z)$ converge?

Suppose $|g(z)| < \frac{C}{|z|^\alpha}$ for some $\alpha, C>0$ and $|z|> \underline{R_0} > 0$.



$\# \{(m,n)\in\mathbb{Z} \mid R\le |z+m\omega_1+n\omega_2| < R+1\} \approx C'\cdot R$

$$|f(z)| \le \sum_{\substack{R=0\\R\in\mathbb{Z}}}^{\infty} \sum_{\substack{m,n\in\mathbb{Z}\\ R\le|z+m\omega_1+n\omega_2|<R+1}} |g(z+m\omega_1+n\omega_2)| \le \left(\sum_{R=0}^{R_0} \cdots\right) + \sum_{R_0}^{\infty} \frac{C}{R^\alpha}\cdot C'\cdot R$$

$\Rightarrow$ For $\alpha>2$, the series $\sum_{m,n\in\mathbb{Z}} g(z+m\omega_1+n\omega_2)$ converges absolutely $\forall z$.

e.g. Let $g(z)=\frac{1}{(z-\alpha)(z-\beta)(z-\gamma)}$, $\alpha,\beta,\gamma\in\mathbb{C}$.

$\Rightarrow$ $f(z)=\sum_{m,n\in\mathbb{Z}} g(z+m\omega_1+n\omega_2)$ is a mero. elliptic fcn.

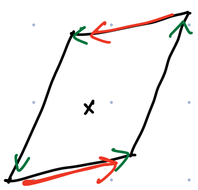
f has 3 poles in the fundamental domain.

Q: Does there exist a mero. elliptic fcn w/ exactly 1 or 2 poles in the fundamental domain? (counted w/ multiplicity)

Fact: $\nexists$ mero. ell. fcn. w/ exactly 1 simple pole in the F.D.

pf: f: mero ell. fcn (of Λ) $\rightsquigarrow$ $\mathbb{C}/\Lambda \xrightarrow{f} \mathbb{C}P^1$ holo.

f has exactly 1 (simple) pole $\Rightarrow$ $\deg(f) = 1$. $\Rightarrow$ f is isom. $\times$. $\square$

pf: • 

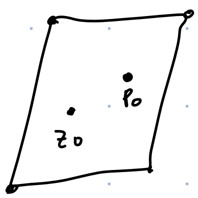
$$\frac{1}{2\pi i}\int_{\square} \frac{f'}{f}\,dz = \#(\text{zeros of } f) - \#(\text{poles of } f)$$

$\parallel$

0

$\Rightarrow$ f has exactly 1 simple pole p_0 & exactly 1 simple zero z_0 in the F.D.

• $$\underbrace{\frac{1}{2\pi i}\int_{\square} z\cdot\frac{f'(z)}{f(z)}\,dz}_{\boxed{\Lambda} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2} = \left(\sum \text{zeros of } f\right) - \left(\sum \text{poles of } f\right).$$

• 

$z_0 - p_0 \in \Lambda$ $\times$. $\square$

Weierstrass $\wp$-function (a mero. ell. fcn. w/ 2 poles in the F.D.)

Idea: • $$\sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^2} = \sum_{m,n\in\mathbb{Z}} \frac{1}{(z+m\omega_1+n\omega_2)^2}$$ doesn't work (doesn't converge)

("λ^{-2}" is not enough to make it converge)

- Consider $\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} = \frac{\lambda^2 - (z+\lambda)^2}{(z+\lambda)^2 \cdot \lambda^2} = \frac{-z^2 - 2z\lambda}{(z+\lambda)^2 \lambda^2}$.

  This has degree -3 in λ.

Def: $$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$$

Claim: $\wp(z)$ is elliptic function (w.r.t. Λ)

pf:
- $\wp'(z)$ is elliptic. $\Rightarrow$ $\wp'(z) = \wp'(z+\omega_1) = \wp'(z+\omega_2)$
- $\wp(z) - \wp(z+\omega_1)$ is a const. fcn, say $\wp(z) - \wp(z+\omega_1) = C$.
- $\wp(z)$ is even. $\Rightarrow$ $C = \wp\left(\frac{-\omega_1}{2}\right) - \wp\left(\frac{\omega_1}{2}\right) = 0$.

  $\Rightarrow$ $\wp(z) = \wp(z+\omega_1)$.
- Similarly, we have $\wp(z) = \wp(z+\omega_2)$. $\square$

Ex: $$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$$
$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)\, \tilde{E}_{2n+2}(\Lambda)\, z^{2n}$$ is the Laurent series exp. at $z=0$

where $\tilde{E}_{2k}(\Lambda) := \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^{2k}}$ ($\approx$ Eisenstein series)

Fact: $$\wp'(z,\Lambda)^2 = 4\wp(z,\Lambda)^3 - 60\tilde{E}_4(\Lambda)\wp(z,\Lambda) - 140\tilde{E}_6(\Lambda).$$

pf:
$$\wp = \frac{1}{z^2} + 3\tilde{E}_4 z^2 + 5\tilde{E}_6 z^4 + 7\tilde{E}_8 z^6 + \cdots$$
$$\wp' = \frac{-2}{z^3} + 6\tilde{E}_4 z + 20\tilde{E}_6 z^3 + \cdots$$

$$\wp'^2 = \boxed{\frac{4}{z^6} - \frac{24\,\tilde{E}_4}{z^2}} - \boxed{80\,\tilde{E}_6} + \cdots$$

$$\wp^3 = \frac{1}{z^6} + \frac{9\,\tilde{E}_4}{z^2} + 15\,\tilde{E}_6 + \cdots$$

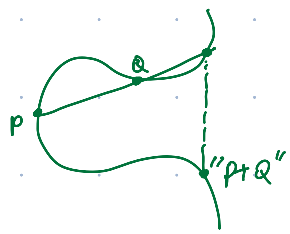
$$\underbrace{\wp'^2 - 4\wp^3 + 60\,\tilde{E}_4\,\wp}_{\text{holo. ell. fcn.} \,\Downarrow\, \text{Const. fcn.}} = \underbrace{-140\,\tilde{E}_6 + \square z^2 + \cdots}_{= -140\,\tilde{E}_6}$$

$\square$

Rmk: $\boxed{\mathbb{C}/\Lambda} \xrightarrow[\cong]{\Phi} \{[x,y,z]\in\mathbb{CP}^2 \mid y^2 z = 4x^3 - 60\,\tilde{E}_4(\Lambda)\,x z^2 - 140\,\tilde{E}_6(\Lambda)\,z^3\}$

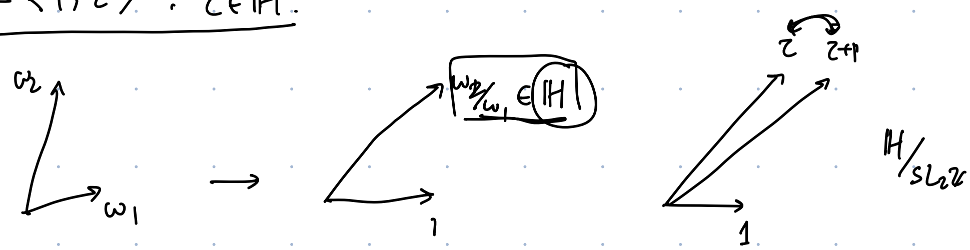
$w \notin \Lambda \longmapsto [\wp(w), \wp'(w), 1]$.

$w = 0 \longmapsto [0, 1, 0]$.



is a group isom.

Ex: $\Lambda = \langle 1, \tau\rangle$, $\tau \in \mathbb{H}$.



- $\wp(z,\tau) = \frac{1}{z^2} + 3\,\boxed{\tilde{E}_4(\tau)}\,z^2 + 5\,\boxed{\tilde{E}_6(\tau)}\,z^4 + \cdots$
- $\wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\,\wp(z,\tau)$ (use the def^n of $\wp(z)$) for $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2\mathbb{Z}$

$$\Longrightarrow \tilde{E}_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k}\,\tilde{E}_{2k}(\tau) \quad \forall k \geq 2.$$

Rmk: $\wp(z)$ has a double pole in any F.D.

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda_-} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$$

What are the zeros of $\wp$?

Eichler-Zagier: Suppose $\Lambda = \langle 1, \tau \rangle$, $\tau \in \mathbb{H}$.

The zeros of $\wp(z,\tau)$ in the F.D. are:

$$\frac{1}{2} \pm \left(\frac{\log(5+2\sqrt{6})}{2\pi i} + 144\pi i \sqrt{6} \int_{\tau}^{\infty} (\sigma - \tau) \frac{E_4(\sigma)^3}{E_6(\sigma)^{3/2}\, j(\sigma)} \, d\sigma \right)$$

Thm Any mero. ell. fcn. can be written as a rational polynomial in $\wp$, $\wp'$.

pf: • $f(z)$ mero. ell. fcn.

• $f(z) = \underbrace{\left(\frac{f(z)+f(-z)}{2} \right)}_{\text{even. ell.}} + \underbrace{\left(\frac{f(z)-f(-z)}{2} \right)}_{\text{odd. ell.}}$

• $\wp'$ odd ell. fcn. $\Rightarrow$ it suffices to consider $\boxed{\text{even mero. ell. fcn. } f}$

• Suppose $z_i \neq 0$ are poles of f in the F.D. of order n_i

then $\underline{f(z)} \cdot \underline{\prod_i (\wp(z) - \wp(z_i))^{n_i}}$ only has pole at 0 in the F.D.

• $f(z) = \boxed{\frac{a_{-2n}}{z^{2n}}} + \cdots\cdots$ Laurent series exp. at $z=0$.

• $\underline{f(z) - a_{-2n}\wp(z)^n} = \frac{*}{z^{2n-2}} + \cdots\cdots$ $\underline{\text{even ell.}}$

• $f(z) - \left(a_{-2n}\wp(z)^n + \cdots + * \wp(z)\right) = \boxed{\text{even. holo. ell.}} = \text{constant}$. $\square$

§ Modular functions & modular forms.

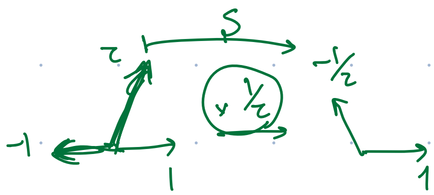
$\downarrow$

$f : \mathbb{H} \to \mathbb{C}$ satisfies $f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)$ $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$.

Rmk:
- $-\mathbb{I}_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \curvearrowright \mathbb{H}$ trivially

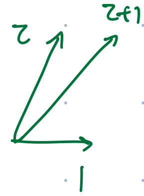
  reduce to the action $PSL_2\mathbb{Z} = SL_2\mathbb{Z}/\{\pm \mathbb{I}\} \curvearrowright \mathbb{H}$.
- $PSL_2\mathbb{Z} = \langle S, T \mid S^2 = (ST)^3 = \mathbb{I} \rangle$.

  where: $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



$\downarrow$

$\tau \mapsto \tau + 1$



$\Lambda_\tau = \langle 1, \tau \rangle$

$\shortparallel$

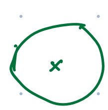
$\Lambda_{\tau+1} = \langle 1, \tau+1 \rangle$

- f is a modular fcn on $\mathbb{H}$ $\iff$ $\boxed{f(\tau) = f(\tau+1) = f\left(\frac{-1}{\tau}\right) \quad \forall \tau \in \mathbb{H}.}$

Rmk: The "simplest" modular fcn. is the j-function. (any mero. modular fcn is a rational polynomial in $j(\tau)$).

$$j(\tau) = \frac{\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, q^n\right)^3}{q \prod_{n=1}^{\infty} (1-q^n)^{24}} \quad \text{where} \quad q = e^{2\pi i \tau}.$$

$\sigma_3(n) = \sum_{d \mid n} d^3$

$\mathbb{H} \longrightarrow$  $D_1^{\times}(0)$

$\tau \longmapsto q = e^{2\pi i \tau}$

$\underline{\operatorname{Im}\tau \to +\infty} \rightsquigarrow \underline{q \to 0}$

Def: $k \geq 0$. A holo. fcn. $f: \mathbb{H} \to \mathbb{C}$ is a <u>modular form of weight k</u>

if
- $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \forall \tau \in \mathbb{H}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$.
- $f(\tau)$ is bounded as $\mathrm{Im}(\tau) \to +\infty$.

Rmk: The 1^{st} condition is equivalent to:
- $f(\tau+1) = f(\tau)$
- $f(-1/\tau) = \tau^k f(\tau)$.

Rmk: For f with $f(\tau+1) = f(\tau)$, it's convenient to introduce $q = e^{2\pi i \tau}$.
(f can be regarded as a fcn in q)

$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & \mathbb{D}_1^\times(0) \\ \tau & \longmapsto & e^{2\pi i \tau} = q \\ \mathrm{Im}\,\tau \to +\infty & \rightsquigarrow & q \to 0. \end{array}$$

" f is bounded near $\mathrm{Im}\,\tau \to +\infty$ " $\iff$ " $f(q)$ can be extended to a holo. fcn on $\mathbb{D}_1(0) = \{q \in \mathbb{C} \mid |q| < 1\}$ ".

Rmk: Consider $f(\tau)\,d\tau$ on $\mathbb{H}$. Is it invariant under $SL_2\mathbb{Z}$-action?

$$f\left(\frac{a\tau+b}{c\tau+d}\right) d\left(\frac{a\tau+b}{c\tau+d}\right) = f\left(\frac{a\tau+b}{c\tau+d}\right) \frac{d\tau}{(c\tau+d)^2}$$

$\Rightarrow$ $f(\tau)d\tau$ is $SL_2\mathbb{Z}$-invar. form $\iff$ $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 f(\tau)$

$\iff$ f is modular form of wt 2.

Similarly, $f(\tau)(d\tau)^{\otimes k}$ is $SL_2\mathbb{Z}$-invar. $\iff$ f is modular fcn of wt $2k$.

Ex/Fact: • If k is odd, then the only modular form of wt k is the zero function.

• If f_i is modular form of weight k_i $(i=1,2)$, then $f_1 \cdot f_2$ is a modular form of wt k_1+k_2

$\Rightarrow$ Modular forms form a graded ring.

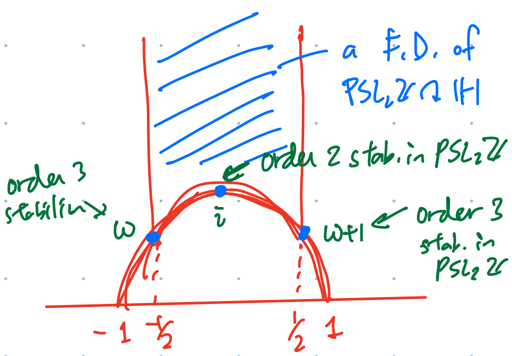
e.g. $\tilde{E}_4(z), \tilde{E}_6(z), \dots$ modular forms of wt $4, 6, \dots$

normalize $\{$ $\{$

$E_4(z), E_6(z) \dots$ $\quad$ $\underline{E_4(q)}, \underline{E_6(q)}, \dots$

$= 1 + O(q)$

Valence formula: $f: \mathbb{H} \to \mathbb{C}$ (nonzero) modular form of weight k.



$$\underset{\substack{\uparrow \\ \text{order of zero} \\ \text{of } f \text{ at } q=0.}}{v_\infty(f)} + \frac{1}{2} \underset{\substack{\uparrow \\ \text{order of zero} \\ \text{of } f \text{ at } i}}{\underline{v_i(f)}} + \frac{1}{3} v_\omega(f) + \underset{\substack{\uparrow \\ \text{sum over orbits of } SL_2\mathbb{Z} \curvearrowright \mathbb{H} \\ \text{except the orbits of } i \text{ and } \omega}}{\underline{\sum_{z \in \mathbb{H}/SL_2\mathbb{Z} \setminus \{[i],[\omega]\}} v_z(f)}} = \frac{k}{12}.$$

<u>Coro:</u> Let $M_k := \{\text{modular forms of weight } k\}$.

- <u>$k<0$:</u> $M_k = \{0\}$.
- <u>$k=0$:</u> $M_0 = \{\text{constant functions}\}$.

  ($f \in M_0 \Rightarrow V_z(f) = 0 \quad \forall z \in \mathbb{H} \cup \{\infty\}$

  Consider $g \equiv f(2i)$. $f-g$ is still a modular form of wt 0, with a zero at $2i$.

  $\longrightarrow f-g \equiv 0. \Rightarrow f$ is const fn.)

- <u>$k=2$:</u> $M_2 = \{0\}$
- <u>$k=4$:</u> $M_4 = \langle E_4 \rangle$. ← $V_\omega(f) = 1$ with no other zeros

  ($f \in M_4 \Rightarrow V_\omega(f) = 1$ with no other zeros.

  For any other $g \in M_4$, we have $f/g \in M_0 \Rightarrow f \equiv (\text{const.}) g$.

  $\Rightarrow M_4$ is 1-dim'l.)

- <u>$k=6$:</u> $M_6 = \langle E_6 \rangle$. ← $V_i(f) = 1$ with no other zeros.
- <u>$k=8$:</u> $M_8 = \langle E_8 \rangle$ ← $V_\omega(f) = 2$ with no other zeros.

  <u>Coro:</u> $E_8 = E_4^2$. ($\Rightarrow$ relation between σ_3, σ_7)

- <u>$k=10$:</u> $M_{10} = \langle E_{10} \rangle$ ← $V_i(f) = V_\omega(f) = 1$ with no other zeros

  <u>Coro:</u> $E_{10} = E_4 E_6$ ($\Rightarrow$ relation among $\sigma_3, \sigma_7, \sigma_9$)

Lemma: $M_{12} = \langle E_4^3, E_6^2 \rangle$ is 2-dim'l.

pf: • E_4^3, E_6^2 are linearly independent. (Follows from the zeros of E_4, E_6).

• $\Delta := \dfrac{E_4^3 - E_6^2}{1728} \in M_{12}$

$= q + O(q^2)$

$E_4^3 = 1 + O(q)$
$E_6^2 = 1 + O(q)$
$E_4^3 - E_6^2 = 1728\, q + O(q^2)$.

has a zero of order 1 at $q=0$. ($\underline{v_\infty(\Delta) = 1}$)

$\Rightarrow$ Δ has no zeros in $\mathbb{H}$.

• For $f \in M_{12}$. $f(q) = a_0 + a_1 q + a_2 q^2 + \cdots$

$\Rightarrow$ $\underline{f(q) - a_0 E_4^3} = * q + O(q^2) \in M_{12}$.

$\Rightarrow$ $f(q) - a_0 E_4^3 = (\text{const.})\,\Delta$

$\Rightarrow$ $f \in \langle E_4^3, E_6^2 \rangle$. $\square$

Thm. $\bigoplus_{k \in \mathbb{Z}} M_k = \mathbb{C}[E_4, E_6]$.

pf. Prove by induction in k. We've proved the cases $\underline{k \leq 12}$.

• Let $f \in M_k$, $k > 12$.

• Chose $a, b \in \mathbb{Z}_{\geq 0}$ s.t. $4a + 6b = k$.

• $f - f(\infty) E_4^a E_6^b \in M_k$ has a zero at ∞.

$\Rightarrow$ $\dfrac{f - f(\infty) E_4^a E_6^b}{\Delta} \in M_{k-12}$. can be written as a poly in E_4, E_6. by inductive hypothesis.

$\square$

<u>Def</u> $j(z) := \dfrac{E_4(z)^3}{\Delta(z)} = \dfrac{1728\, E_4(z)^3}{E_4(z)^3 - E_6(z)^2}$.

- $j(z)$ is invariant under the action $SL_2\mathbb{Z} \curvearrowright \mathbb{H}$.
- $j(z)$ has a simple pole at ∞ $(q=0)$, with no poles in $\mathbb{H}$.

  $\left(j(q) = \frac{1}{q} + 744 + 196884 q + \cdots \right)$
- j has zero of order 3 at (the orbit of) ω, no other zeros.
- $j(z) - 1728$ has zero of order 2 at (the orbit of) i, no other zeros

<u>Thm</u> Any mero. modular function is a rational poly. of $j(z)$.

pf: • f is mero. modular fcn. with poles at $\underline{p_1, \ldots, p_k \in \mathbb{H}}$ of order $n_1, \ldots, n_k$.

Then: $f(z) \cdot \prod_{i=1}^{k} (j(z) - j(p_i))^{n_i}$ is a mero. modular fcn. with pole only at ∞. $(q=0)$

- Suppose $f(q) = \underline{\underline{a_{-n}}} q^{-n} + \cdots\cdots$
- $f(q) - a_{-n} j(q)^n = \dfrac{*}{q^{n-1}} + \cdots\cdots$ mero modular fcn.
- $\underbrace{f(q) - \left(* j(q)^n + * j(q)^{n-1} + \cdots + * j(q) \right)}_{\text{modular fcn}} = \underbrace{* + * q + \cdots}_{\text{holo. fcn in } q} \equiv C.$

modular form of wt 0

$\Downarrow$

const. fcn.

$\square$

Rmk: Say 2 integral positive-definite quadratic form $D := b^2 - 4ac < 0$

$$Q(x,y) = ax^2 + bxy + cy^2, \quad Q'(x,y) = a'x^2 + b'xy + c'y^2$$

are <u>equivalent</u>: if they're the same up to a $SL_2\mathbb{Z}$-change of variable.

(e.g. $x' = x+y$, $y' = y$).

Easy Fact: • "equivalent" $\Longleftrightarrow$ " $D = D'$ and $j\left(\frac{b+\sqrt{-D}}{2a}\right) = j\left(\frac{b'+\sqrt{-D'}}{2a'}\right)$ ".

• For each $D<0$, $\exists$ <u>finitely many</u> equiv. class w/ disc. D.

$\rightsquigarrow$ a finite set of j-value for each $D<0$.

Hard Fact: • For each $D<0$, $\displaystyle\prod_{\substack{[Q] \\ \mathrm{disc}(Q)=D}} \left(x - j([Q])\right) \in \mathbb{Z}[x]$.

• When there is only one equiv. class w/ disc. D.

($\Longleftrightarrow D = -1, -2, -3, -7, -11, -19, -43, -67, -163$)

the j-value of the quad. form is an integer.

e.g. $Q(x,y) = x^2 + xy + 41y^2$. $D = -163$.

$$\Rightarrow \boxed{j\left(\frac{1+\sqrt{-163}}{2}\right) \in \mathbb{Z}}$$

$$j(q) = \frac{1}{q} + 744 + 196884\,q + \cdots\cdots$$

$$j(z) = \frac{1}{e^{2\pi i z}} + 744 + 196884\, e^{2\pi i z} + \cdots\cdots$$

$$\mathbb{Z} \ni j\left(\frac{1+\sqrt{-163}}{2}\right) = \frac{1}{e^{-\pi\sqrt{163}}} + 744 + \underbrace{196884\, e^{-\pi\sqrt{163}} + \cdots\cdots}_{\text{small}}$$

$$\approx e^{\pi\sqrt{163}} + 744 + \underbrace{\qquad}_{\text{small}} \qquad \Rightarrow e^{\pi\sqrt{163}} \approx \mathbb{Z}.$$