§ Sheaves:

Def: X. top space. A <u>presheaf</u> of abelian groups on X is a pair $(\mathcal{F}, \rho)$ consisting of:

- a family $\{\mathcal{F}(\mathcal{U})\}_{\underset{\text{open}}{\mathcal{U} \subseteq X}}$ of abel. gps.
- a family $\{\rho^{\mathcal{U}}_{V}\}_{V \subseteq \mathcal{U}}$ of gp homom.

with:
- $\rho^{\mathcal{U}}_{\mathcal{U}} = id_{\mathcal{F}(\mathcal{U})}$ $\forall\, \mathcal{U} \subseteq X$.
- $\rho^{V}_{W} \circ \rho^{\mathcal{U}}_{V} = \rho^{\mathcal{U}}_{W}$ $\forall\, W \subseteq V \subseteq \mathcal{U}$.

Rmk:
- We usually simply write it as $\mathcal{F}$.,
- The homom. $\rho^{\mathcal{U}}_{V}$ are called <u>restrictions</u>, instead of $\rho^{\mathcal{U}}_{V}(f)$ for $f \in \mathcal{F}(\mathcal{U})$, we just write $f|_{V} := \rho^{\mathcal{U}}_{V}(f) \in \mathcal{F}(V)$.
- Similarly, one can define presheaves of rings, modules, ...

<u>e.g.</u> X: R.S. $\mathcal{C}^{\infty}(\mathcal{U}) := \{\text{smooth fcns } \mathcal{U} \to \mathbb{R}\}$, $\rho =$ restriction.

$\mathcal{O}(\mathcal{U}) := \{\text{holo. fcns } \mathcal{U} \to \mathbb{C}\}$

Def: A presheaf $\mathcal{F}$ is called a <u>sheaf</u> if $\forall\, \mathcal{U} \subseteq X$, and $\forall\, \mathcal{U}_i \subseteq \mathcal{U}$ w/ $\mathcal{U} = \cup \mathcal{U}_i$.

1) If $f, g \in \mathcal{F}(\mathcal{U})$ satisfies $f|_{\mathcal{U}_i} = g|_{\mathcal{U}_i}$ $\forall i$, then $f = g$.

2) If $\{f_i \in \mathcal{F}(\mathcal{U}_i)\}$ has the property that $f_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = f_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$ $\forall i, j$. then $\exists\, f \in \mathcal{F}(\mathcal{U})$ s.t. $f|_{\mathcal{U}_i} = f_i$ $\forall i$.

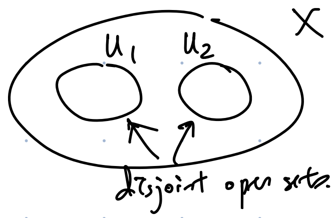
is unique by 1)

e.g. X top space, G abel. gp.

Define a presheaf $\mathcal{F}$ on X as: $\mathcal{F}(U) := \begin{cases} G & \text{if } U \neq \phi \\ 0 & \text{if } U = \phi \end{cases}$

and $\rho^U_V = id_G$ if $V \neq \phi$, $\rho^U_V = 0$ if $V = \phi$

It's not a sheaf in general:



pick $g_1 \neq g_2$ in G.

$g_1 \in \mathcal{F}(U_1)$, $g_2 \in \mathcal{F}(U_2)$, $g_1|_{\underset{\phi}{\underset{\|}{U_1 \cap U_2}}} = 0 = g_2|_{U_1 \cap U_2}$.

But $\nexists$ $g \in \mathcal{F}(U_1 \cup U_2)$ st. $g|_{U_1} = g_1$ and $g|_{U_2} = g_2$.

<u>one can modify it to become a sheaf.</u> ("sheafification")

$\tilde{\mathcal{F}}(U) := \{$locally constant $f: U \to G\}$.

<u>Stalk of a (pre)sheaf $\mathcal{F}$ at $a \in X$:</u>

Define an equivalence relation $\sim$ on $\coprod_{a \in U} \mathcal{F}(U)$:

$f \in \mathcal{F}(U) \sim g \in \mathcal{F}(V)$ if $\exists$ $a \in W \subseteq U \cap V$ st. $f|_W = g|_W$.

$$\mathcal{F}_a := \coprod_{a \in U} \mathcal{F}(U) \Big/ \sim \quad \left(= \varinjlim_{a \in U} \mathcal{F}(U) \right)$$

$*$ $\forall a \in U$, we have $\rho_a : \mathcal{F}(U) \longrightarrow \mathcal{F}_a$

$f \longmapsto \rho_a(f)$: "<u>germ</u>" of f at a.

e.g. $\mathcal{O}_a \longleftrightarrow$ convergent power series in $z-a$

§ Čech cohomology

Notation:

- X: top. space, $\mathcal{U} = \{U_i\}_{i \in J}$. open cover of X.
- $\mathcal{F}$: sheaf (of abel. gps). on X.
- For $I = (i_0, \dots, i_p)$ a $(p+1)$-tuple of elements of J. let:
  - $U_I := U_{i_0 \cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$
  - $U_{i_0 \cdots \hat{i_j} \cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_{j-1}} \cap U_{i_{j+1}} \cap \cdots \cap U_{i_p}$.
  - There is an inclusion: $\delta_j^p : U_{i_0 \cdots i_p} \hookrightarrow U_{i_0 \cdots \hat{i_j} \cdots i_p}$.

Def: Čech p^{th}-cochain gp. of $\mathcal{F}$:

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{(i_0, \dots, i_p) \in J^{p+1}} \mathcal{F}(U_{i_0 \cdots i_p}).$$

coboundary operators: $\delta^p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$.

defined by: $\delta^p := \sum_{j=0}^{p+1} (-1)^j \underline{\underline{\mathcal{F}(\delta_j^{p+1})}}$.

$$\mathcal{F}(U_{i_0 \cdots \hat{i_j} \cdots i_p}) \xrightarrow{\text{res.}} \mathcal{F}(U_{i_0 \cdots i_p})$$

e.g.

$$\begin{array}{ccccc}
C^0(\mathcal{U}, \mathcal{F}) & \xrightarrow{\delta^0} & C^1(\mathcal{U}, \mathcal{F}) & \xrightarrow{\delta^1} & C^2(\mathcal{U}, \mathcal{F}) \\
\| & & \| & & \| \\
\prod_{j \in J} \mathcal{F}(U_j) & \longrightarrow & \prod_{(i,j) \in J^2} \mathcal{F}(U_i \cap U_j) & \longrightarrow & \prod_{(i,j,k) \in J^3} \mathcal{F}(U_{ijk}) \\
\Psi & & \Psi & & \\
(f_j)_{j \in J} & \longmapsto & (g_{i,j}) & \longmapsto & (h_{ijk}) \\
 & & \| & & \| \\
 & & f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j} & & g_{jk}|_{U_{ijk}} - g_{ik}|_{U_{ijk}} + g_{ij}|_{U_{ijk}}.
\end{array}$$

Ex: $\delta^{p+1} \circ \delta^p = 0 \ \forall p \geq 0$, i.e. this defines a complex.

Def: p-th Čech cohomology:

Čech p-th cocycles

$$H^p(\mathcal{U}, \mathcal{F}) := \frac{Z^p(\mathcal{U}, \mathcal{F})}{B^p(\mathcal{U}, \mathcal{F})} := \frac{\ker \delta^p}{\operatorname{Im} \delta^{p-1}}$$

Čech p-th coboundaries

$$0 \to C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^0} C^1(\mathcal{U}, \mathcal{F}) \to \cdots$$

e.g. $H^0(\mathcal{U}, \mathcal{F}) = \ker\left(C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^0} C^1(\mathcal{U}, \mathcal{F})\right)$

- $(f_j)_{j \in J} \in H^0(\mathcal{U}, \mathcal{F}) \iff f_j|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} \ \forall i, j$.
- Since $\mathcal{F}$ is a sheaf, $\exists!\ f \in \mathcal{F}(X)$ s.t. $f|_{U_i} = f_i \ \forall i$.

$\Rightarrow$ $H^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)$, and is independent of the choice of $\mathcal{U}$.

- Define: $H^0(X, \mathcal{F}) := \mathcal{F}(X)$. (global sections of $\mathcal{F}$).

Def: Let $\mathcal{U} = \{U_i\}_{i \in I}$, $\mathcal{V} = \{V_j\}_{j \in J}$ be two open coverings of X.
Say $\mathcal{V}$ is a refinement of $\mathcal{U}$. (denoted $\mathcal{U} < \mathcal{V}$) if $\exists\ \tau : J \to I$ s.t. $V_j \subseteq U_{\tau(j)} \ \forall j \in J$.

In this case ($\mathcal{U} < \mathcal{V}$), can define

$$\tau^p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{V}, \mathcal{F})$$
$$f \longmapsto \tau^p f$$

where $(\tau^p f)_{j_0 \cdots j_p} := f_{\tau(j_0) \cdots \tau(j_p)}|_{V_{j_0 \cdots j_p}}$.

Ex:

$$\begin{tikzcd}
{} \arrow[r] & C^p(\mathcal{U},\mathcal{F}) \arrow[r, "\delta^p"] \arrow[d, "\tau^p"'] \arrow[dr, phantom, "\circlearrowright"] & C^{p+1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\tau^{p+1}"] & {} \\
{} \arrow[r] & C^p(\mathcal{V},\mathcal{F}) \arrow[r, "\delta^p"'] & C^p(\mathcal{V},\mathcal{F}) \arrow[r] & {}
\end{tikzcd}$$

$\rightsquigarrow \quad H^p(\mathcal{U},\mathcal{F}) \xrightarrow{\tau^{*p}} H^p(\mathcal{V},\mathcal{F})$ for $\mathcal{U} < \mathcal{V}$.

Lemma: Suppose $\mathcal{U} < \mathcal{V}$, and if both $\tau_1 : J \to I$, $\tau_2 : J \to I$

satisfy: $V_j \subseteq U_{\tau_1(j)}$ and $V_j \subseteq U_{\tau_2(j)}$ $\forall j \in J$.

then: $\tau_1^{*p} = \tau_2^{*p}$ $\forall p \geq 0$.

pf:

$$\begin{tikzcd}
{} \arrow[r] & C^p(\mathcal{U},\mathcal{F}) \arrow[r, "\delta^p"] \arrow[d, shift right, "\tau_1^p"'] \arrow[d, shift left, "\tau_2^p"] \arrow[dl, "k^p"] & C^{p+1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, shift right, "\tau_1^{p+1}"'] \arrow[d, shift left, "\tau_2^{p+1}"] \arrow[dl, "k^{p+1}"] & {} \\
{} \arrow[r] & C^p(\mathcal{V},\mathcal{F}) \arrow[r, "\delta^p"'] & C^p(\mathcal{V},\mathcal{F}) \arrow[r] & {}
\end{tikzcd}$$

Idea: construct a chain homotopy..

For $f \in C^p(\mathcal{U},\mathcal{F})$, define $k^p f \in C^{p-1}(\mathcal{V},\mathcal{F})$ where:

$$(k^p f)_{j_0 \dots j_{p-1}} := \sum_{h=0}^{p-1} (-1)^h f_{\tau_1(j_0)\dots\tau_1(j_h)\tau_2(j_h)\dots\tau_2(j_{p-1})}\Big|_{V_{j_0\dots j_{p-1}}}$$

Ex: $\delta^{p-1} \circ k^p + k^{p+1} \circ \delta^p = \tau_2^p - \tau_1^p$ on $C^p(\mathcal{U},\mathcal{F})$.

Ex: $\Longrightarrow \tau_1^{*p} = \tau_2^{*p}$

This gives a well-defined homom.: $H^p(\mathcal{U},\mathcal{F}) \longrightarrow H^p(\mathcal{V},\mathcal{F})$ for $\mathcal{U}<\mathcal{V}$.

Def: <u>p-th Čech cohomology</u>: is defined to be the <u>direct limit</u> runs through all open covers of X:

$$H^p(X,\mathcal{F}) := \varinjlim_{\mathcal{U}} H^p(\mathcal{U},\mathcal{F})$$

$$= \coprod_{\mathcal{U}} H^p(\mathcal{U},\mathcal{F}) \Big/ \sim$$

where $\xi_1 \in H^p(\mathcal{U}_1,\mathcal{F}) \sim \xi_2 \in H^p(\mathcal{U}_2,\mathcal{F})$ if $\exists\, \mathcal{V}$ st. $\mathcal{U}_1<\mathcal{V}$, $\mathcal{U}_2<\mathcal{V}$

and

$$\begin{tikzcd} H^p(\mathcal{U}_1,\mathcal{F}) \arrow[dr] & \\ & H^p(\mathcal{V},\mathcal{F}) \\ H^p(\mathcal{U}_2,\mathcal{F}) \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \xi_1 \arrow[r, mapsto] & \cdot \arrow[d, phantom, "\shortparallel"] \\ \xi_2 \arrow[r, mapsto] & \cdot \end{tikzcd}$$

$\{\mathcal{U}_i\}$ $\{V_j\}$. $\tau: J \to I$

Lemma: For $\mathcal{U}<\mathcal{V}$, the homom. $H^1(\mathcal{U},\mathcal{F}) \longrightarrow H^1(\mathcal{V},\mathcal{F})$ is injective.

pf:

$$\begin{tikzcd} 0 \arrow[r] & \prod \mathcal{F}(\mathcal{U}_{i_0}) \arrow[r] & \prod \mathcal{F}(\mathcal{U}_{i_0 i_1}) \arrow[r] & \prod \mathcal{F}(\mathcal{U}_{i_0 i_1 i_2}) \\ & & (f_{i_0 i_1}) \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\ & (g_{j_0}) \arrow[r, mapsto] & (f_{j_0 j_1}) & \\ 0 \arrow[r] & \prod \mathcal{F}(V_{j_0}) \arrow[r] & \prod \mathcal{F}(V_{j_0 j_1}) \arrow[r] & \prod \mathcal{F}(V_{j_0 j_1 j_2}) \end{tikzcd}$$

- $f_{i_1 i_2} - f_{i_0 i_2} + f_{i_0 i_1} = 0$ on $\mathcal{U}_{i_0 i_1 i_2}$.
- $f_{j_0 j_1} := f_{\tau(j_0)\tau(j_1)}|_{V_{j_0 j_1}}$
- $f_{j_0 j_1} = g_{j_1} - g_{j_0}$ on $V_{j_0 j_1}$

$$\Rightarrow \quad g_{j_1} - g_{j_0} = f_{j_0 j_1} = f_{\tau(j_0)\tau(j_1)} \quad \text{on } V_{j_0 j_1} \subseteq U_{\tau(j_0)\tau(j_1)}$$

$$\Rightarrow \quad g_{j_1} - g_{j_0} = -f_{i\tau(j_1)} + f_{i\tau(j_0)} \quad \text{on } V_{j_0} \cap V_{j_1} \cap U_i \subseteq U_{\tau(j_0)\tau(j_1)i}.$$

$$\Rightarrow \quad \underbrace{f_{i\tau(j_1)} + g_{j_1}}_{\text{on } U_i \cap V_{j_1}} = \underbrace{f_{i\tau(j_0)} + g_{j_0}}_{\text{on } U_i \cap V_{j_0}} \quad \text{on } V_{j_0} \cap V_{j_1} \cap U_i.$$

$$\Rightarrow \quad \exists\, h_i \in \mathcal{F}(U_i) \text{ s.t. } h_i|_{V_j} = f_{i\tau(j)} + g_j \quad \forall j.$$

Then, on $U_{i_0} \cap U_{i_1} \cap V_j$, we have:

$$f_{i_0 i_1} = f_{i_0 \tau(j)} + f_{\tau(j) i_1} = (h_{i_0} - g_j) + (g_j - h_{i_1}) = h_{i_0} - h_{i_1}.$$

This holds for every j, thus: $f_{i_0 i_1} = h_{i_0} - h_{i_1}$ on $U_{i_0} \cap U_{i_1}$.

$$\Rightarrow \quad (-h_i) \xmapsto{\delta^0} (f_{i_0 i_1}). \quad \square$$

<u>Coro:</u> $H^1(X, \mathcal{F}) = 0$ iff. $H^1(\mathcal{U}, \mathcal{F}) = 0$ $\forall$ open cover $\mathcal{U}$ of X.

<u>e.g.</u> Consider the locally constant sheaf $\underline{\mathbb{Z}}$. To get a nontrivial elt. of $H^1(X, \underline{\mathbb{Z}})$, it suffices to find an open cover $\mathcal{U}$ st. $H^1(\mathcal{U}, \underline{\mathbb{Z}}) \neq 0$.
For instance, suppose $U_1, \ldots, U_n$ are <u>connected</u> open subsets where $U_i \cap U_{i+1} \neq \phi$, $U_1 \cap U_n \neq \phi$, and other pairwise intersections are empty.
(so X is "coarsely" a circle.).

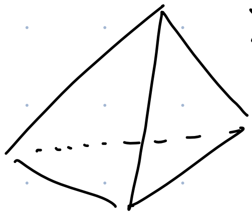
$$0 \to \prod \underline{\mathbb{Z}}(U_i) \longrightarrow \underbrace{\prod \underline{\mathbb{Z}}(U_{i_0 i_1})}_{(f_{12}, f_{23}, \ldots, f_{n1}) \,\cong\, \mathbb{Z}^n} \longrightarrow \prod \underline{\mathbb{Z}}(\overset{\phi}{\overset{\shortparallel}{U_{i_0 i_1 i_2}}}) = 0$$

$$(f_i) \longmapsto (f_{i_0 i_1} = f_{i_1} - f_{i_0})$$

$$\text{Image} = \{ f_{12} + \cdots + f_{n1} = 0 \}.$$

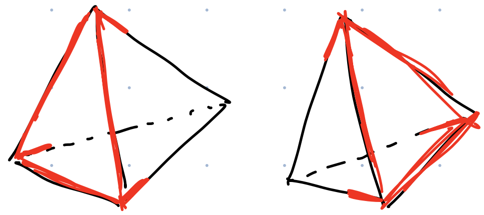
Rmk: Refinement does <u>not</u> induce injective map $H^p(\mathcal{U},\mathcal{F}) \longrightarrow H^p(\mathcal{V},\mathcal{F})$ for $p \geq 2$. in general.

e.g.

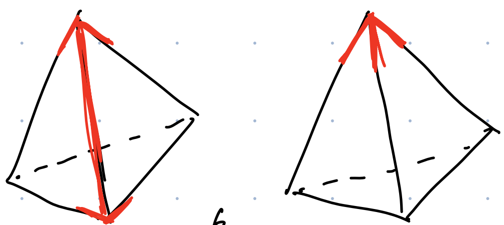


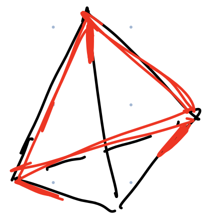
X = 1-skeleton of a 3-simplex.

$\mathcal{U} = \{\mathcal{U}_1, \cdots, \mathcal{U}_4\}$.



$$\prod \mathbb{Z}(\mathcal{U}_{ij}) \xrightarrow{\delta^1} \prod \mathbb{Z}(\mathcal{U}_{ijk}) \xrightarrow{\delta^2} \overset{0}{\overset{\|}{\mathbb{Z}(\underset{\overline{\phi}}{\mathcal{U}_{1234}})}}$$





$$(f_{12}, f_{23}, f_{34}, f_{13}, f_{24}, f_{14}) \in \mathbb{Z}^6 \longmapsto (\underset{123}{f_{23} - f_{13} + f_{12}}, \underset{124}{-(f_{24} - f_{14} + f_{12})}, \underset{134}{f_{34} - f_{14} + f_{13}}, -(f_{34} - f_{24} + f_{23}))$$

sum = 0.

$\Longrightarrow H^2(\mathcal{U}, \mathbb{Z}) \cong \mathbb{Z} \neq 0.$

But, we know that $H^2(X, \mathbb{Z}) = 0$

(one can refine $\mathcal{U} < \mathcal{V}$ so that $\mathcal{V}$ has no 3-fold intersections)

Def:

- An open set $U \subseteq X$ is <u>acyclic for $\mathcal{F}$</u> if $H^p(U,\mathcal{F})=0$ $\forall p>0$.
- A covering $\mathcal{U}=\{U_i\}_{i\in J}$ is a <u>Leray covering for $\mathcal{F}$</u> if U_I is acyclic for $\mathcal{F}$ $\forall$ indices $I \subsetneq J$. (i.e. $H^k(U_{i_0\cdots i_p},\mathcal{F})=0$ $\forall k>0, \forall i_0\cdots i_p$)

<u>Thm</u> (Leray): If $\mathcal{U}$ is a Leray covering for $\mathcal{F}$, then $H^p(X,\mathcal{F}) \cong H^p(\mathcal{U},\mathcal{F})$. $\forall p$

<u>Thm</u> (Leray). $\mathcal{U}=\{U_i\}_{i\in I}$ open cover of X. Suppose $H^1(U_i,\mathcal{F})=0$ $\forall i\in I$.

Then: $H^1(X,\mathcal{F}) \cong H^1(\mathcal{U},\mathcal{F})$.

Leray cover of 1st order for $\mathcal{F}$.

pf: It suffices to show that $\forall$ finer covering $\mathcal{U}<\mathcal{V}$, $I \xleftarrow{\tau} J$, the injective map

$H^1(\mathcal{U},\mathcal{F}) \longrightarrow H^1(\mathcal{V},\mathcal{F})$ is actually isomorphism.

$$
\begin{array}{ccccccc}
0 \longrightarrow & \prod \mathcal{F}(U_{i_0}) & \longrightarrow & \prod \mathcal{F}(U_{i_0 i_1}) & \longrightarrow & \prod \mathcal{F}(U_{i_0 i_1 i_2}) \\
 & & & (F_{i_0 i_1}) & \longmapsto & 0 \\
 & & & & & \\
 & (h_j) & & (f_{j_0 j_1}) & \longmapsto & 0 \\
 & & & \in & & \\
0 \longrightarrow & \prod \mathcal{F}(V_{j_0}) & \longrightarrow & \prod \mathcal{F}(V_{j_0 j_1}) & \longrightarrow & \prod \mathcal{F}(V_{j_0 j_1 j_2})
\end{array}
$$

- $f_{j_1 j_2} - f_{j_0 j_2} + f_{j_0 j_1} = 0$ on $V_{j_0 j_1 j_2}$.
- <u>Goal:</u> Find $(F_{i_0 i_1}) \mapsto 0$, (h_j), s.t.

  $F_{\tau(j_0)\tau(j_1)} - f_{j_0 j_1} = h_{j_1} - h_{j_0}$ on $V_{j_0 j_1}$.

- Since $H^1(U_i,\mathcal{F})=0$. Consider $\{U_i \cap V_j\}_{j\in J}$ open cover of U_i.

  $\Longrightarrow$ $H^1(U_i \cap \mathcal{V}, \mathcal{F}) = 0$.

- $(f_{j_0j_1})$ when restricting on $\mathcal{U}_i$, should be coboundary, i.e.

  $\exists\, g_{ij} \in \mathcal{F}(\mathcal{U}_i \cap V_j)$ s.t. $f_{j_0j_1} = -g_{ij_1} + g_{ij_0}$ on $\mathcal{U}_i \cap V_{j_0} \cap V_{j_1}$.

- On $\mathcal{U}_{i_0} \cap \mathcal{U}_{i_1} \cap V_{j_0} \cap V_{j_1}$, $\Rightarrow g_{i_1j_1} - g_{i_0j_1} = g_{i_1j_0} - g_{i_0j_0}$

$$-g_{i_0j_1} + g_{i_0j_0} = f_{j_0j_1} = -g_{i_1j_1} + g_{i_1j_0}.$$

$\Rightarrow \exists\, F_{i_0i_1} \in \mathcal{F}(\mathcal{U}_{i_0} \cap \mathcal{U}_{i_1})$ s.t. $F_{i_0i_1} = g_{i_1j} - g_{i_0j}$ on $\mathcal{U}_{i_0} \cap \mathcal{U}_{i_1} \cap V_j$.

- Easy to check that $\delta^1((F_{i_0i_1})) = 0$.
- On $V_{j_0j_1}$,

$$F_{\tau(j_0)\tau(j_1)} - f_{j_0j_1} = (g_{\tau(j_1)j_0} - g_{\tau(j_0)j_0}) - (g_{\tau(j_1)j_1} + g_{\tau(j_1)j_0})$$
$$= g_{\tau(j_1)j_1} - g_{\tau(j_0)j_0}.$$

- Take $h_j = g_{\tau(j)j}$ on V_j. $\square$

Rmk: X: real n-dim'l mfd. Fact: Any open cover admits a refinement s.t. its $(n+2)$-fold intersection is empty. $\Rightarrow H^p(X, \mathcal{F}) = 0 \quad \forall p > n$.

Thm: $\mathcal{F}$: sheaf of $\mathcal{C}^\infty$-fcns on X, or more generally, sheaf of modules over the sheaf of $\mathcal{C}^\infty$-fcns.
mfld w/ a partition of unity

(e.g. sheaves of smooth forms $A^0, A^{1,0}, A^{0,1}, A^2, \dots$)

Then $H^p(X, \mathcal{F}) = 0 \quad \forall p > 0$.

pf (For $p=1$): Claim: $H^1(\mathcal{U}, \mathcal{F}) = 0 \;\; \forall$ open cover $\mathcal{U}$ of X.

- Choose $\{\rho_i\} \subseteq \mathcal{C}^\infty(X)$ with $K_i := \operatorname{supp} \rho_i \subseteq U_i$, where $\{K_i\}$ is a locally finite cover of X. st $\sum \rho_i(x) = 1$.

-
$$\prod \mathcal{F}(U_i) \xrightarrow{\delta^0} \prod \mathcal{F}(U_i \cap U_j) \xrightarrow{\delta^1} \prod \mathcal{F}(U_i \cap U_j \cap U_k)$$
$$(f_{ij}) \longmapsto 0$$

- $f_{jk} - f_{ik} + f_{ij} = 0$ in U_{ijk},

  $f_{ii} = 0$ (set $j=k=i$), $\quad f_{ij} = -f_{ji}$ on U_{ij} (set $k=i$)

- Goal: Find (g_i) s.t. $f_{ij} = g_j - g_i$.

- How to get from $(f_{ij} \in \mathcal{F}(U_i \cap U_j))$ to $(g_i \in \mathcal{F}(U_i))$?

* ρ_j supports in U_j, so $\rho_j f_{ij}$ can be extended by 0 $\rightsquigarrow$ $\mathcal{F}(U_i)$

well-defined, since $\mathcal{F}$ is a sheaf of $\mathcal{C}^\infty$-module

- Define $g_i := \sum_k \rho_k f_{ik}$ Then:

$$g_i - g_j = \sum_k \rho_k (f_{ik} - f_{jk}) = \sum_k \rho_k f_{ij} = f_{ij}. \quad \square$$

Ex: Prove this for all $p \geq 1$.

(e.g. $p=2$: $g_{ab} := \sum_c \rho_c f_{abc}$.)

e.g. X : R.S. $\pi_1(X) = 0$.

Then $H^1(X, \mathbb{C}) = H^1(X, \mathbb{Z}) = 0$.

pf: • Let $(c_{ij}) \in Z^1(\mathcal{U}, \mathbb{C})$ By Thm, $\exists\, (g_i) \in \overset{\text{0-cochain}}{\overset{\downarrow}{C^0}}(\mathcal{U}, \mathcal{E}_{\mathbb{C}}^{\infty})$

s.t. $c_{ij} = g_i - g_j$. on U_{ij}

• Then $dg_i = dg_j$ on U_{ij}.

$\Rightarrow \exists\, \omega \in A^1(X)_{\mathbb{C}}$ s.t. $\omega|_{U_i} = dg_i$.

• $d\omega = 0$, since $\pi_1(X) = 0$, we have $\omega = df$ for some $f \in \mathcal{E}^{\infty}(X)_{\mathbb{C}}$.

• On U_i, $\omega = df = dg_i \Rightarrow b_i := g_i - f$ is locally const.

$\Rightarrow (b_i) \in C^0(\mathcal{U}, \mathbb{C})$

$c_{ij} = g_i - g_j = b_i - b_j$. $\square$

pf: • Let $(a_{ij}) \in Z^1(\mathcal{U}, \mathbb{Z})$, By the first part, $\exists\, (b_i) \in C^0(\mathcal{U}, \mathbb{C})$

s.t. $a_{ij} = b_i - b_j$. on U_{ij}.

$\Rightarrow \exp(2\pi i b_i) = \exp(2\pi i b_j)$ on U_{ij}.

$\Rightarrow \exists\, c \in \mathbb{C}$ s.t. $\exp(2\pi i c) = \exp(2\pi i b_i)$ on U_i

$\Rightarrow d_i := b_i - c \in \mathbb{Z}$

• On U_{ij}, $a_{ij} = b_i - b_j = d_i - d_j$. $\square$