§ Hodge decomp. of cpt. Riemannian mflds.

- The Riemannian metric g assigns to each $p\in M$ a positive definite sym. bilinear form $g_p: T_pM\times T_pM \to \mathbb{R}$, varying smoothly in $p\in M$.

  $\rightsquigarrow$ induces an inner product on $\Lambda^k T_p^* M$.

- The Hodge $*$-operator: $\quad *: \Lambda^k T^*M \longrightarrow \Lambda^{n-k} T^*M \quad (n=\dim M)$

  is uniquely defined by $\langle \alpha, \beta\rangle = \int_M \alpha\wedge(*\beta)$

<u>Facts</u>:

- $*^2 = (-1)^{(n-k)k}$ on $\Lambda^k T_p^* M$

- Let $d^*: A^*(M) \longrightarrow A^{*-1}(M)$ be the adjoint operator of d (w.r.t. g.) i.e.

  $\langle \alpha, d^*\beta\rangle = \langle d\alpha, \beta\rangle$.

  Then, for $\alpha\in A^k(M)$ and $\beta\in A^{k+1}(M)$, we have:

$$\langle \alpha, d^*\beta\rangle = \langle d\alpha, \beta\rangle$$

$$= \int_M d\alpha\wedge *\beta = \int_M d(\alpha\wedge *\beta) - (-1)^k \alpha\wedge d*\beta$$

$$d(\alpha\wedge *\beta) = d\alpha\wedge *\beta + (-1)^k \alpha\wedge d(*\beta)$$

$$= (-1)^{k+1}\int_M \alpha\wedge d*\beta \quad \left(= **(d*\beta)(-1)^{(n-k)k}\right)$$

$$= (-1)^{k+1+nk-k^2}\langle \alpha, *d*\beta\rangle$$

$$\Longrightarrow \quad d^* = (-1)^{nk+1} * d * .$$

<u>Def</u>: The Hodge-Laplace operator: $\Delta := (d+d^*)^2 = dd^* + d^*d$

- $\langle \Delta\alpha, \alpha\rangle = \langle dd^*\alpha + d^*d\alpha, \alpha\rangle = \|d^*\alpha\|^2 + \|d\alpha\|^2$

  Thus, "$\Delta\alpha = 0$" $\Longleftrightarrow$ "$d\alpha = d^*\alpha = 0$".

Hodge decomposition thm: M: cpt. Riemannian mfd.

Let $\mathbb{H}^p(M) := \{\alpha \in A^p(M) \mid \Delta\alpha = 0\}$ harmonic p-forms.

1) $\dim \mathbb{H}^p(M) < +\infty$.

2) $A^p(M) = \mathbb{H}^p(M) \oplus^{\perp} \Delta(A^p(M))$, i.e. $\operatorname{Im}\Delta = \mathbb{H}^p(M)^{\perp}$.

Coro. Any class $[\alpha] \in H^p(M)$ has a unique harmonic repn:

- choose any repn. α. $\rightsquigarrow$ $\alpha = \underset{\uparrow \atop \text{harmonic}}{\beta} + \underbrace{\Delta\gamma}_{= d(d^*\gamma) + d^*(d\gamma)}$

$$d\alpha = 0 \Rightarrow \underline{dd^*(d\gamma)} = 0$$

$$\Rightarrow \|d^*d\gamma\|^2 = \langle d^*d\gamma, d^*d\gamma\rangle = \langle \underbrace{dd^*d\gamma}_{0}, d\gamma\rangle = 0$$

$$\Rightarrow d^*d\gamma = 0.$$

$$\Rightarrow \alpha = \beta + d(d^*\gamma)$$

$\Rightarrow$ $[\alpha]$ admits harmonic repn.

- uniqueness of harmonic repn: Suppose $\underbrace{\alpha_1 - \alpha_2}_{\text{both harmonic}} = d\beta$.

$$\|d\beta\|^2 = \langle d\beta, d\beta\rangle = \langle d^*\underset{\uparrow \atop \text{harmonic}}{d\beta}, \beta\rangle = \langle 0, \beta\rangle = 0. \Rightarrow d\beta = 0. \quad \square$$

2 Fundamental thms in elliptic PDE: (Apply to $\boxed{\Delta}$)

Regularity: Any weak sol$^{\text{n}}$ $\Delta\omega = \beta$ is automatically smooth

Compactness: For $\{\alpha_n\} \subseteq H$, if $\|\alpha_n\|, \|\Delta\alpha_n\| \le C$. both bdd,

then $\{\alpha_n\}$ has a Cauchy subseq.

pf 1). Suppose $\dim \mathbb{H}^p = \infty$. Then $\exists$ $\underline{u_1, \dots, u_n, \dots} \in \mathbb{H}^p$, $\|u_i\| = 1$,
$\Delta u_i = 0$, $u_i \perp u_j$.

$\{u_i\}$ has no Cauchy subseq. Contradict w/ Compactness Thm!

2) It's clear that $\mathrm{Im}\,\Delta \subseteq \mathbb{H}^\perp$.

$$\Big(\langle \Delta\alpha, \underset{\substack{\uparrow \\ \Delta\beta = 0}}{\beta} \rangle = \langle \alpha, \Delta\beta \rangle = 0. \Big)$$

Conversely: For $\alpha \in \mathbb{H}^\perp$. $\boxed{\underline{\text{Want}}:\ \text{Find } w \text{ s.t. } \Delta w = \alpha}$

(Similar to what we did last time:
"$\alpha \in \mathbb{H}^\perp$" $\iff$ "$\int_X \beta = 0$"

- Find weak sol$^{\underline{n}}$ using boundedness + Riesz repn.
- Show it must be smooth using regularity.)

"$\Delta w = \alpha$" $\longleftrightarrow$ "$\langle \phi, \Delta w \rangle = \langle \phi, \alpha \rangle$" $\forall \phi$.
$$\langle \phi, \Delta w \rangle = \langle \Delta\phi, \underset{\substack{\uparrow \\ \text{weak sol}^{\underline{n}}}}{w} \rangle$$

Define a functional $l_\alpha : \mathrm{Im}\,\Delta \longrightarrow \mathbb{R}$
$$\Delta\phi \longmapsto \langle \phi, \alpha \rangle.$$

- $\underline{l_\alpha \text{ is well-defined}}$: If $\Delta\phi = \Delta\phi'$, then $\phi - \phi' \in \mathbb{H}$,
$\Rightarrow \langle \phi - \phi', \alpha \rangle = 0$ since $\alpha \in \mathbb{H}^\perp$.

To apply Riesz repn, we need $\underline{l_\alpha \text{ is bounded}}$ $|\langle \phi, \alpha \rangle| \overset{?}{\leq} C \cdot \|\Delta\phi\|$.

Claim: $\exists\, C>0$ s.t. $\|\beta\| \leq C\cdot \|\Delta\beta\| \quad \forall \beta\in \mathbb{H}^\perp$.

Assuming the Claim. Then:

$$|\ell_\alpha(\Delta\phi)| = |\langle \alpha, \phi\rangle| = |\langle \alpha, \phi - \underset{\text{the projection of } \phi \text{ onto } \mathbb{H}}{H(\phi)}\rangle| \leq \|\alpha\|\cdot \|\underbrace{\phi - H(\phi)}_{\text{in } \mathbb{H}^\perp}\|$$

$$\leq \|\alpha\|\cdot C\cdot \|\underbrace{\Delta(\phi - H(\phi))}_{=\Delta\phi}\| \implies \ell_\alpha \text{ is bounded.}$$

- Riesz repn thm $\Rightarrow$ $\exists\, w$ (in the completion of $A^p(M)$)

  s.t. $\langle \Delta\phi, w\rangle = \ell_\alpha(\Delta\phi) = \langle \phi, \alpha\rangle$.

  i.e. w is a weak sol$^{\underline{n}}$ of $\Delta w = \alpha$.

- Regularity thm of ell. op. $\Rightarrow$ $w\in A^p(M)$ and $\Delta w = \alpha$.

  $\Rightarrow \alpha \in \operatorname{Im}\Delta$. $\square$

---

pf: Suppose the contrary. Then $\exists\, \beta_j \in \mathbb{H}^\perp$ st. $\|\beta_j\|=1$, $\|\Delta\beta_j\| \to 0$.

Compactness thm $\Rightarrow$ $\{\beta_j\}$ has a Cauchy subseq., (may assume it's Cauchy).

Define $\ell(\psi) := \lim_{j\to\infty} \langle \beta_j, \psi\rangle$; clearly a bounded functional, and

$$\ell(\Delta\psi) = \lim_{j\to\infty} \langle \beta_j, \Delta\psi\rangle = \lim_{j\to\infty} \langle \Delta\beta_j, \psi\rangle = 0.$$

$\Rightarrow$ ℓ is a weak sol$^{\underline{n}}$ of $\Delta\beta = 0$.

By Regularity thm, $\exists\, \beta\in A^p$ st. $\ell(\psi) = \langle \beta, \psi\rangle\ \forall \psi$, and $\Delta\beta = 0$ ($\beta \in \mathbb{H}$)

In particular, $\ell(\beta) = \lim_{j\to\infty} \underbrace{\langle \underset{\in \mathbb{H}^\perp}{\beta_j}, \underset{\in \mathbb{H}}{\beta}\rangle}_{=0} = \underset{=\ell(\beta)}{\langle \beta, \beta\rangle} \Rightarrow \beta = 0$ $\quad$ $\times$ with $\|\beta_j\| = 1$. $\square$

Thm X: cpt. R.S. w/ $g(X) \geq 2$. Then $|\mathrm{Aut}(X)| \leq 84(g-1)$

- We've proved that any finite subgp $G \subseteq \mathrm{Aut}(X)$ has order $\leq 84(g-1)$. (by Riemann-Hurwitz on $X \longrightarrow X/G$)

  It remains to show: $\boxed{|\mathrm{Aut}(X)| < +\infty}$

Sketch:

- $G \curvearrowright H^*(X, \mathbb{C})$ preserving the Hodge decomp, and lattice $H^1(X, \mathbb{Z})$.

  $\Longrightarrow$ $\rho: G \longrightarrow GL(H^{1,0}(X))$ with discrete image $\rho(G)$

- Moreover, $\rho(G)$ preserves the inner product $\|\alpha\|^2 := \int_X i\alpha \wedge \bar{\alpha} > 0$. on $H^{1,0}(X)$. $\Longrightarrow$ $\rho(G) \subseteq \underset{\substack{\uparrow \\ \text{unitary gp.} \Rightarrow \text{compact}}}{U}(H^{1,0}(X)) \subseteq GL(H^{1,0}(X))$

  $\Longrightarrow$ $|\rho(G)| < +\infty$.

- $\rho: G \longrightarrow GL(H^{1,0}(X))$ is injective:
  - Suppose $f: X \longrightarrow X$ holo. auto, acts trivially on $H^{1,0}(X)$.
  - $|\mathrm{Fix}(f)| < +\infty$: Otherwise, since $\mathrm{Fix}(f)$ is a closed cpx subvar. of X $\Rightarrow$ $\dim_{\mathbb{C}} \mathrm{Fix}(f) = 1$ $\Rightarrow$ $\mathrm{Fix}(f) = X$.
  - Lefschetz fixed pt thm:
    - All fixed point of f has 'degree' 1
    - $\sum_{x \in \mathrm{Fix}(f)} \mathrm{ind}_x(f) = \sum_k (-1)^k \mathrm{tr}(H^k \xrightarrow{f^*} H^k) = 1 - 2\mathrm{Tr}(\mathrm{id}_{H^1}) + 1 = 2 - 2g < 0$ $\cancel{\ast}$.

Uniformization Thm: X: conn. simply conn. non-cpt. R.S.

$\Rightarrow X \cong \mathbb{C}$ or $\mathbb{H}$.

Coro: Any (conn.) R.S. is isom. to either:

- $\mathbb{CP}^1$
- $\mathbb{C}$, $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times$, $\mathbb{C}/\Lambda$.
- $\mathbb{H}/\Gamma$. where $\Gamma \subseteq PSL(2,\mathbb{R})$ discrete, acts freely on $\mathbb{H}$.

(Follows directly from classificatn of holo. auto. of $\mathbb{CP}^1$, $\mathbb{C}$, $\mathbb{H}$).

Def: X: non-cpt. $\phi: X \longrightarrow \mathbb{R}$, $c \in \mathbb{R}$.

We say ϕ tends to c at infinity in X if

$\forall \varepsilon > 0$, $\exists \underset{\text{cpt}}{K} \subseteq X$ st. $|\phi(x) - c| < \varepsilon \quad \forall x \in X \setminus K$.

We say ϕ tends to $+\infty$ at infinity in X if

$\forall A \in \mathbb{R}$, $\exists \underset{\text{cpt}}{K} \subseteq X$ st. $\phi(x) > A \quad \forall x \in X \setminus K$.

Thm: X: conn. simply conn. non-cpt. R.S.

If ρ is a real 2-form of cpt. supp on X w/ $\int_X \rho = 0$.

then $\exists \phi: X \longrightarrow \mathbb{R}$ s.t.
- $\Delta \phi = \rho$ on X
- ϕ tends to 0 at infinity in X.

Claim: Thm $\Longrightarrow$ Uniformization Thm:

- Choose a pt. $p \in X$ and local coord. z.,
  As before, let β be a smooth cut-off fcn, $=1$ near p.
  Let $A := \bar{\partial}\left(\frac{\beta}{z}\right)$. $\leftarrow$ (0,1)-form. supported in an annulus around p.
- Let $\rho := \partial A$. Then $\int_X \rho = 0$. (ρ has cpt. supp.)
  By Thm (apply to real & imaginary parts of ρ),
  $\exists\ g: X \longrightarrow \mathbb{C}$ s.t. • $\partial\bar{\partial} g = \rho$,
  • $\mathrm{Re}(g), \mathrm{Im}(g) \longrightarrow 0$ at infinity of X.
- Let $a := \overset{(0,1)}{(A - \bar{\partial} g)} + \overline{\overset{(1,0)}{(A - \bar{\partial} g)}}$ a real 1-form.
  By construction, we have $\partial a = \bar{\partial} a = 0$. $\Rightarrow$ $da = 0$.
- X is simply connected, i.e. $\pi_1(X) = 0$ $\Rightarrow$ $H_1(X, \mathbb{Z}) \cong \pi_1(X)^{ab} = 0$.

$\Longrightarrow$ By universal coeff. thm, $H^1(X, \mathbb{R}) \cong \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{R}) = 0$.

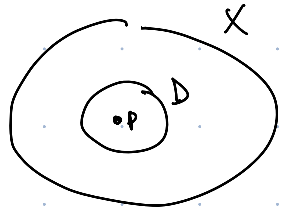
$$H^1(X,\mathbb{R}) \cong H^1_{dR}(X).$$

$\Rightarrow$ $\exists\ \psi: X \longrightarrow \mathbb{R}$ s.t. $a = d\psi$.

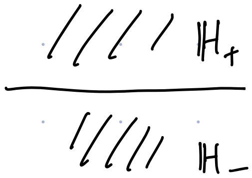
- Thus, $A = \bar{\partial} g + \bar{\partial}\psi$. $\leftarrow$ (0,1)-part of $d\psi = a$.

$\Rightarrow$ $\bar{\partial}\left(\frac{\beta}{z} - (g + \psi)\right) = 0$ on $X \setminus \{p\}$

$\Rightarrow$ $f := g + \psi$ is mero. on X w/ simple pole at p,
and $\mathrm{Im}(f) \longrightarrow 0$ at infinity of $X \setminus D$



$\rightsquigarrow$ $f : X \longrightarrow \mathbb{CP}^1$. holo.

• 

$= \mathbb{C} \subseteq \mathbb{CP}^1$.

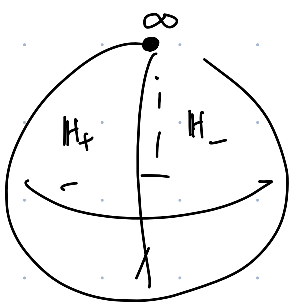
Let $X_\pm := f^{-1}(\mathbb{H}_\pm)$.

• The holo. maps $f_\pm : X_\pm \longrightarrow \mathbb{H}_\pm$ are <u>proper</u>. :

Suppose $B \underset{cpt}{\subseteq} \mathbb{H}_+$. $\Rightarrow$ $\exists\, \varepsilon > 0$ st.
$\mathrm{Im}(z) > \varepsilon \;\forall z \in B$.

The fact that $\mathrm{Im}(f) \longrightarrow 0$ at infinity $\Rightarrow$ $f^{-1}B$ is <u>cpt</u>.

• f has simple pole at p $\Rightarrow$ f is a local homeom. near $p \mapsto \infty$.



$\Rightarrow$ $X_\pm$ are nonempty.

Since $f_\pm : X_\pm \to \mathbb{H}_\pm$ are proper, there is a well-defined <u>degree</u> of $f_\pm$.

<u>Claim</u> $\deg(f_\pm) = 1$. ($\Rightarrow$ $X_\pm \hookrightarrow \mathbb{H}_\pm$)

pf: • $\mathrm{Im}(f) \to 0$ at infinity $\Rightarrow$ $\exists\, K \underset{cpt}{\subseteq} X$ st. $\mathrm{Im}(f)(x) < 1 \;\; \forall x \notin K$.

• Suppose $\deg(f_+) \geq 2$. Then, $\forall n \geq 1$, $\exists\, x_n, \tilde{x}_n \in X_+$
st. $f(x_n) = f(\tilde{x}_n) = in$, and either $x_n \neq \tilde{x}_n$ or $x_n = \tilde{x}_n$ and $f'(x_n) = 0$.

- All $\{x_n\}, \{\tilde{x}_n\} \subseteq K$. $\Rightarrow$ $\exists$ conv. subseq.

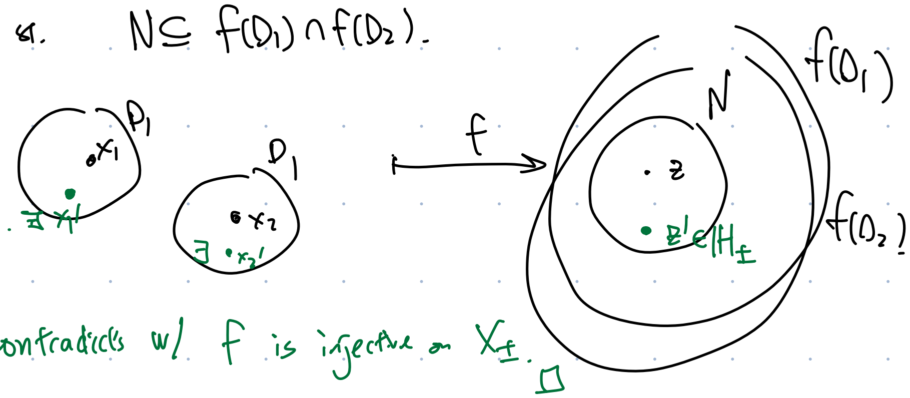
  $\Rightarrow$ they must conv. to p.

  This contradicts w/ f is local homeom. near p. $\square$

<u>Claim.</u> $f: X \hookrightarrow \mathbb{CP}^1$ is injective.

- Suppose $f(x_1) = f(x_2) = z \in \mathbb{CP}^1$.
- $\exists$ disjoint open discs D_1, D_2 of x_1, x_2, and nbhd $z \in N \subseteq \mathbb{CP}^1$.

  st. $N \subseteq f(D_1) \cap f(D_2)$.



Contradicts w/ f is injective on X_f. $\square$

$\Rightarrow$ X is isom. to an open subset of $\mathbb{C}$ ($X \neq \mathbb{CP}^1$ since X non-cpt.)

w/ $\pi_1(X) = 0$.

1) $X \cong \mathbb{C}$

2) $X \cong$ proper, open, simply conn. subset of $\mathbb{C}$ $\overset{\text{Rieman's thm.}}{\Longrightarrow}$ $X \cong \mathbb{H}$ $\square$

§ Sheaves:

Def: X. top space. A <u>presheaf</u> of abelian groups on X is a pair $(\mathcal{F}, \rho)$ consisting of:

- a family $\{\mathcal{F}(U)\}_{U \underset{\text{open}}{\subseteq} X}$ of abel. gps.
- a family $\{\rho^U_V\}_{V \subseteq U}$ of gp homom.

with:
- $\rho^U_U = \mathrm{id}_{\mathcal{F}(U)}$ $\forall U \subseteq X$.
- $\rho^V_W \circ \rho^U_V = \rho^U_W$ $\forall W \subseteq V \subseteq U$.

Rmk:
- We usually simply write it as $\mathcal{F}$.,
- The homom. ρ^U_V are called <u>restrictions</u>, instead of $\rho^U_V(f)$ for $f \in \mathcal{F}(U)$, we just write $f|_V := \rho^U_V(f) \in \mathcal{F}(V)$.
- Similarly, one can define presheaves of rings, modules, ...

<u>e.g.</u> X: R.S. $\mathcal{C}^\infty(U) := \{\text{smooth fcns } U \to \mathbb{R}\}$, ρ = restriction.

$\mathcal{O}(U) := \{\text{holo. fcns } U \to \mathbb{C}\}$

Def: A presheaf $\mathcal{F}$ is called a <u>sheaf</u> if $\forall U \subseteq X$, and $\forall U_i \subseteq U$ w/ $U = \cup U_i$.

1) If $f, g \in \mathcal{F}(U)$ satisfies $f|_{U_i} = g|_{U_i}$ $\forall i$, then $f = g$.

2) If $\{f_i \in \mathcal{F}(U_i)\}$ has the property that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ $\forall i, j$. then $\exists f \in \mathcal{F}(U)$ s.t. $f|_{U_i} = f_i$ $\forall i$.