Riemann-Roch formula (not the most general form, we'll discuss sheaf, sheaf cohomology, ... to develop necessary tools later)

- $p_1, \ldots, p_d \in X$, $D := \{p_1, \ldots, p_d\} \subseteq X$.
- $H^0(D) := \{$ mero. fcn. on X w/ at worst simple poles at $D\}$.
- $H^0(K-D) := \{$ holo. 1-forms on X that vanish at each $p_i\}$.

Then: $$h^0(D) - h^0(K-D) = d - g + 1.$$

- recall that residue is not well-defined for mero fcn, but is well-defined for mero. 1-forms. Given $f \in H^0(D)$, at each $p_i \in D$, consider

$$\begin{aligned} T^*_{p_i}X &\longrightarrow \mathbb{C} \\ \omega &\longmapsto \mathrm{res}_{p_i}(f\omega) \end{aligned}$$

which can be considered as an element of $T_{p_i}X$.

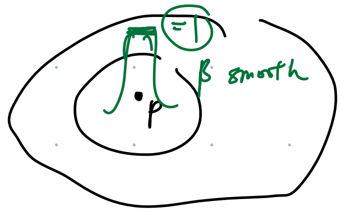
$$\Longrightarrow \quad \mathrm{Res}: H^0(D) \longrightarrow \bigoplus_{i=1}^{d} T_{p_i}X$$

- $\mathrm{Ker}(\mathrm{Res}) = \{$ holo. fcns on $X\} = \{$ constant fcns. $\}$.
- The image of Res is the local residue data at D that can be realized by a global mero. fcn. on X.

Recall that, this is closely related to $H^{0,1}(X)$, which gives the obstruction of such local data to glue globally.

Let's define a map $T_pX \longrightarrow H^{0,1}(X)$ as follows:

- Let z be a local coord. at p, an element of T_pX is of the form $a\frac{\partial}{\partial z}$.
- Consider a bump fcn, $=1$ near p.
  $=0$ outside of a small disc.



- Then $A := \bar\partial(\beta \cdot \frac{a}{z}) = \underbrace{(\bar\partial\beta)}_{\overset{\shortparallel}{0} \text{ near } p} \cdot \frac{a}{z}$ defines a global $(0,1)$-form.

$\ast$ $\frac{a}{z}$ is indep. of the choice of coord. !!

- $[A] \in H^{0,1}(X)$ is indep. of the choice of β.

This defines a map:
$$\begin{aligned} T_pX &\longrightarrow H^{0,1}(X). \\ a\frac{\partial}{\partial z} &\longmapsto \left[(\bar\partial\beta)\cdot\frac{a}{z}\right] \end{aligned}$$

We have an exact seq.:

$$0 \to \mathbb{C} \xrightarrow{\text{const.}} H^0(D) \xrightarrow{\text{Res}} \bigoplus_{i=1}^{d} T_{p_i}X \xrightarrow{\mathbb{A}} H^{0,1}(X)$$

(the given local residue data in $\oplus T_{p_i}X$ can be repn. by global mero. fcn.
$\Leftrightarrow$ their image in $H^{0,1}(X)$ sum to 0.)

The dual map: $\mathbb{A}^T: H^{0,1}(X)^* \longrightarrow \bigoplus_{i=1}^{d} T^*_{p_i}X$
$$\begin{array}{c} H^{0,1}(X)^* \\ \cong \\ H^{1,0}(X) \end{array}$$

Linear alg. Fact:

$$\underbrace{\dim \operatorname{Ker} A}_{} - \dim \ker A^T = d - \dim H^{0,1}(X) = d - g.$$

$$\dim \operatorname{Ker} A = \dim \operatorname{Im}(\operatorname{Res}) = h^0(D) - 1.$$

<u>Claim:</u> $A^T : H^{1,0}(X) \longrightarrow \bigoplus_{i=1}^{d} T_{p_i}^* X$ is $\underline{(2\pi i)\cdot ev}$,

where $ev: H^{1,0}(X) \longrightarrow \bigoplus T_{p_i}^* X$.

$$\omega \longmapsto (\omega|_{p_i})$$

Assuming the Claim, then

$$\dim \ker A^T = \dim \{\text{holo. 1-forms vanishing at } D\}$$

$$= h^0(K - D),$$

which proves Riemann-Roch.

$$H^{1,0}(X) \cong H^{0,1}(X)^*$$

<u>Pf of Claim:</u> $T_pX \longrightarrow H^{0,1}(X)$, $\quad H^{0,1}(X)^* \longrightarrow T_p^* X$

$$a\frac{\partial}{\partial z} \longmapsto \bar\partial\left(\beta \cdot \frac{a}{z}\right). \qquad \omega \in H^{1,0} \longmapsto \left(T_pX \to \mathbb{C},\ a\frac{\partial}{\partial z} \mapsto \bar\partial\left(\beta \frac{a}{z}\right)\right)$$

$$\omega \updownarrow \left([\gamma] \in H^{0,1} \longmapsto \int \omega \wedge \gamma\right) \qquad \downarrow \int \omega \wedge \bar\partial\left(\beta \cdot \frac{a}{z}\right)$$

$$\int \omega \wedge \bar\partial\left(\beta \frac{a}{z}\right) = a \int \omega \wedge \bar\partial(\beta) \frac{1}{z}$$

$$= a \int_{X \setminus D} \omega \wedge (\bar\partial \beta) \frac{1}{z} = a \int_{X \setminus D} \bar\partial\Big(\underbrace{\omega \wedge \frac{\beta}{z}}_{\text{holo 1-form on } X \setminus D}\Big) = a \int_{X \setminus D} d\left(\omega \wedge \frac{\beta}{z}\right)$$

$\omega = g(z)\,dz$ locally

$$\overset{\text{Stokes}}{=} a \int_\gamma \omega \wedge \frac{\beta}{z} = a \oint_\gamma \frac{\omega}{z} = 2\pi i \cdot a \cdot g(0)$$

$$\mathrm{ev}: H^{1,0}(X) \longrightarrow T_p^* X$$
$$\underset{g(z)dz}{\overset{\shortparallel}{\omega}} \longmapsto \left(\begin{array}{c} T_pX \longrightarrow \mathbb{C} \\ a\frac{\partial}{\partial z} \mapsto a\,g(0) \end{array} \right) . \quad \square$$

Last time:

- $H^{1,0}(X) \cong \overline{H^{0,1}(X)}$; $\alpha \longmapsto [\bar{\alpha}]$.
- $H^{1,0}(X) \cong H^{0,1}(X)^*$
- $H^1_{dR}(X) \cong H^{1,0}(X) \oplus H^{0,1}(X)$; $\quad H^2_{dR}(X) \cong H^{1,1}(X)$.

Rmk: More generally, for a cpt Kähler mfd X, we have the Hodge decomposition

$$H^k_{dR}(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

Our proofs are based on the following thm:

Thm: X: cpt. R.S. $\quad \rho \in A^2(X)_{\mathbb{C}}$.
" $\exists f \in A^0(X)_{\mathbb{C}}$ st. $\Delta f = \rho$ " $\Longleftrightarrow$ " $\int_X \rho = 0$ ".

Hodge thm:
- every class in $H^k_{dR}(M)$ can be uniquely represented by a harmonic k-form (denoted $\mathbb{H}^k(M)$)
- There is an orthogonal decomposition:
$$A^k(M) = \mathbb{H}^k(M) \oplus \Delta(A^k(M))$$

(we'll discuss later!)

In the thm., ($\Rightarrow$) is easy:

$$\int_X \Delta f = 2i \int_X \bar{\partial}\partial f = 2i \int_X d(\partial f) = 0.$$

($\Leftarrow$) is the most crucial part of the thm. It can be directly obtained from Hodge thm:

- $\mathbb{H}^2(X) \cong H^2_{dR}(X) \cong H^2(X)$ is 1-dim'l, spanned by the volume form of X.

  We can repn it by a harmonic 2-form ω. we have $\int_X \omega \neq 0$.

- $\rho \in A^2(X) = \mathbb{H}^2(X) \oplus \Delta(A^2(X))$

  i.e. $\exists\, c \in \mathbb{C},\ f \in A^0(X)$ st. $\rho = c \cdot \omega + \Delta f$.

  $$\Rightarrow \quad 0 = \int_X \rho = c \underbrace{\int_X \omega}_{\neq 0} + \underbrace{\int_X \Delta f}_{= 0}$$

  $\Rightarrow$ $c = 0$. $\square$

<u>Rmk:</u> Further generalization: "Elliptic Diff'l operator" on cpt mfd:

$\mathcal{L}$ w/ adjoint $\mathcal{L}^*$: $\langle \mathcal{L}f, g\rangle = \langle f, \mathcal{L}^* g\rangle$.

$\mathcal{L}$ elliptic $\Rightarrow$ $A^k(M) = \underbrace{\ker \mathcal{L}^*}_{\text{finite dim'l}} \overset{\perp}{\oplus} \mathcal{L} A^k(M)$

$\Delta^* = \Delta \Rightarrow A^k(M) = \mathbb{H}^k(M) \overset{\perp}{\oplus} \Delta A^k(M)$

$\boxed{\bullet \text{ It suffices to prove the } \underline{\text{real}} \text{ version of the main thm.}}$

$\int \rho = 0 \Longrightarrow \exists f$ s.t. $\Delta f = \rho$

Def: $f, g \in \mathcal{C}^\infty(X)$

- Dirichlet inner product:

$$\langle f, g\rangle_D := 2i \int_X \partial f \wedge \bar{\partial} g$$

$$= -\int_X \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right) dx \wedge dy$$

$$= \int_X g \,\Delta f = \int_X f \,\Delta g$$

- Dirichlet norm: $\|f\|_D^2 := \langle f, f\rangle_D = \int_X f \Delta f = \int_X |df|^2.$

- They actually define on the quotient: $\mathcal{C}^\infty(X)/\mathbb{R}$ ← constant functions.

Fact: $(\mathcal{C}^\infty(X)/\mathbb{R}, \langle -,-\rangle_D)$ is a pre-Hilbert space. (not complete)

Let $\rho \in A^2(X)_{\mathbb{R}}$. $\forall \phi, \psi \in \mathcal{C}^\infty(X)$,

$$\int_X \psi(\rho - \Delta\phi) = \int_X \psi\rho - \langle \phi, \psi\rangle_D$$

$$\text{"}\Delta\phi = \rho\text{"} \iff \text{"}\int_X \psi(\rho - \Delta\phi) = 0 \ \forall \psi\text{"}$$

$$\iff \text{"}\int_X \psi\rho = \langle \phi, \psi\rangle_D \ \forall \psi\text{"}$$

Main Thm can be formulated as: If $\int_X \rho = 0$,

(Then $\hat{\rho}: C^\infty(X)/\mathbb{R} \longrightarrow \mathbb{R}$ is well-defined,
$\psi \longmapsto \int_X \psi\rho$)

$$\boxed{\exists\, \phi \in \mathcal{C}^\infty(X) \text{ s.t. } \hat{\rho}(\psi) = \langle \psi, \phi \rangle_D \quad \forall \psi.}$$

Riesz repn thm: Let H be a Hilbert space,
$\vartheta: H \longrightarrow \mathbb{R}$ be a bounded linear map $(|\vartheta(x)| \leq C\|x\|)$
Then: $\exists\, z \in H$ s.t.
$$\vartheta(x) = \langle x, z \rangle \quad \forall x \in H.$$

- Any pre-Hilbert space has a formal completion: a point in the completed Hilbert space is an equiv. class of Cauchy seq. $\{\psi_i\}$, where $\{\psi_i\} \sim \{\psi_i'\}$ if $\|\psi_i - \psi_i'\| \longrightarrow 0$.
- Need 1: For $\int \rho = 0$, the functional $\hat{\rho}: \mathcal{C}^\infty(X)/\mathbb{R} \longrightarrow \mathbb{R}$ is bounded,
$\psi \longmapsto \int \psi\rho$

  i.e. $\exists\, C > 0$ st. $\left|\int \psi\rho\right| \leq C\|\psi\|_D \quad \forall \psi \in \mathcal{C}^\infty(X)$.

- Such functional extends naturally to a functional on H (completion of $\mathcal{C}^\infty(X)/\mathbb{R}$).
- By Riesz repn thm, $\exists\, \phi \in H$ ("weak sol^{n}") st. $\hat{\rho}(\psi) = \langle \psi, \phi \rangle_D \quad \forall \psi$.
- Need 2: For $\int \rho = 0$, such weak sol^{n} is actually an element of $\mathcal{C}^\infty(X)/\mathbb{R}$. $(\subseteq H)$. (regularity of sol^{n} of ell PDE)

Boundedness of $\hat{\rho}$: $\left(\left| \int_X \psi\rho \right| \le C \|\psi\|_D = \int_X |d\psi|^2 \right)$

(Need: Bound the L^2-norm of ψ by the L^2-norm of $d\psi$.)

"L^2-Poincaré ineq." Ω – bdd. convex domain in $\mathbb{R}^2$,

ψ – sm. fcn. on an open set containing $\overline{\Omega}$.

$$\int_\Omega |\psi|^2 d\mu \le \frac{d^2}{\pi^2} \int_\Omega |d\psi|^2 d\mu., \qquad d = \mathrm{diam}.(\Omega).$$

Local case: Suppose ρ is supported on a single coord. chart. $\cong \Omega$.

Then ρ can be identified with a smooth fcn. on Ω.

(Cauchy-Schwarz)

$$|\hat{\rho}(\psi)| = \left|\int_X \rho\psi\right| = \left|\int_\Omega \rho\psi \, d\mu\right| \le \|\rho\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)}$$

$$\le \underbrace{\boxed{\|\rho\|_{L^2(\Omega)} \cdot \frac{\mathrm{diam}(\Omega)}{\pi}}}_{= C} \cdot \|\psi\|_D \leftarrow \|d\psi\|_{L^2(\Omega)} \text{ by def}^{\text{n}} \text{ of Dirichlet norm.}$$

General ρ w/ $\int_X \rho = 0$:

- Since $H^2_{dR}(X) \xrightarrow[\sim]{\int} \mathbb{R}$, $[\rho] = 0$ in H^2_{dR}, i.e. $\rho = d\theta$ for some $\theta \in A^1(X)$.
- Let $\{U_\alpha\}$ be a (finite) cover of X by coord. charts, w/ partition of unity $\{\chi_\alpha\}$ supported on each U_α.
- Then: $\rho_\alpha := d(\chi_\alpha \theta)$ is supp. on U_α, w/ $\int_X \rho_\alpha = 0$.

  $\Rightarrow |\hat{\rho}_\alpha(\psi)| = \left|\int_X \rho_\alpha \psi\right| \le C_\alpha \cdot \|\psi\|_D \quad \forall \psi \in C^\infty(X)$.
- $\hat{\rho}(\psi) = \int_X \rho\psi = \int_X (d\theta)\psi = \int_X d((\Sigma \chi_\alpha)\theta)\psi = \Sigma \int_X \rho_\alpha \psi$ ← finite sum.

  $= \sum_\alpha \hat{\rho}_\alpha(\psi). \Rightarrow |\hat{\rho}(\psi)| \le (\Sigma C_\alpha) \|\psi\|_D$. □

The weak sol^n is actually smooth: $\quad \tilde{\xi}(\psi) = \langle \psi, "\varphi" \rangle$.

- By Riesz repn. thm, we know "φ" exists in the completion of $\mathcal{C}^\infty(X)/\mathbb{R}$,

  i.e. $\exists$ seq. $\{\phi_i\} \subseteq \mathcal{C}^\infty(X)/\mathbb{R}$. Cauchy w.r.t. the Dirichlet norm.

  with: $\langle \phi_i, \psi \rangle \longrightarrow \tilde{\xi}(\psi)$ as $i \to +\infty$.

Claim: $\{\phi_i\}$ conv. to a function which is locally L^2.

- Note that there is a choice $\phi_i + (\text{const.})$ involve, so we need to be a bit more careful.
- Let Ω be a coord. chart. We choose suitable constants s.t. $\int_\Omega \phi_i \, d\mu = 0 \quad \forall i$.
- L^2-Poincaré ineq. $\Rightarrow \quad \|\phi_i - \phi_j\|_{L^2(\Omega)} \leq C(\Omega) \cdot \|\phi_i - \phi_j\|_D$.
- Since $\{\phi_i\}$ is Cauchy is Dirichlet norm, $\{\phi_i\}$ is a Cauchy seq. in $L^2(\Omega)$.

  By completeness of $L^2(\Omega)$, $\{\phi_i\} \longrightarrow \phi \in L^2(\Omega)$.

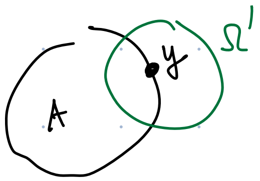
- We claim that the same seq. $\{\phi_i\}$ (with the chosen constants), converges locally in L^2 over all of X. This uses the standard open/closed argument.

$$A := \{ x \in X \mid \exists \text{local chart } \Omega \text{ of } x \text{ s.t. } \phi_i \longrightarrow \phi \in L^2(\Omega) \}.$$

By previous discussion, $A \neq \phi$. It's clear that A is open.

- It remains to show that A is closed. Suppose $y \in \bar{A} \setminus A$.

  We know that $\exists \; y \in \Omega'$ and $\{c_i\}$ st. $\phi_i - c_i \longrightarrow \tilde{\phi} \in L^2(\Omega')$.



For $x \in A \cap \Omega'$, in a small nbhd of x, both $\{\phi_i\}$ & $\{\phi_i - c_i\}$ conv. in L^2

$\Rightarrow$ $c_i \longrightarrow c$ for some c.

$\Rightarrow$ $\{\phi_i\} \longrightarrow \tilde{\phi} + c$ in $L^2(\Omega')$, thus $y \in A$. $\square$

<u>We have so far:</u> $\exists\, \phi$: locally L^2-fcn. on X s.t.

$$\underbrace{\hat{\rho}(\psi)}_{\int_X \psi \cdot \rho} = \underbrace{\langle \psi, \phi \rangle}_{\int_X (\Delta\psi)\cdot\phi} \quad \forall\, \psi \in \mathcal{C}^\infty(X)$$

i.e. a locally L^2-sol$^{\underline{n}}$ of "$\Delta\phi = \rho$".

<u>Want:</u> $\phi \in \mathcal{C}^\infty(X)$.

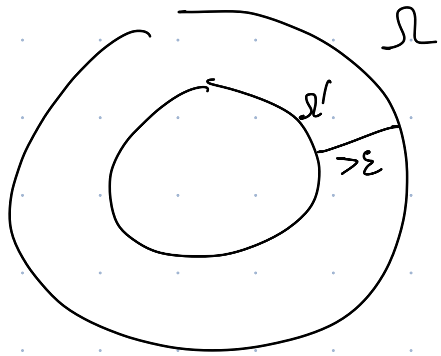
This is a purely local statement of ϕ. So we can just consider coord. charts Ω and sm. fcns supp. on Ω.

<u>Thm:</u> Ω: bdd. open set in $\mathbb{C}$, ρ: sm. 2-form on Ω.

Suppose $\phi \in L^2(\Omega)$ has:

$$\int_\Omega (\Delta\chi)\,\phi = \int_\Omega \chi \cdot \rho \quad \forall\, \chi \in \mathcal{C}_c^\infty(\Omega).$$

Then ϕ is smooth.



It suffices to show the smoothness of ϕ in Ω'.

Let ρ' be a smooth fcn. st. $\rho' = \rho$ in Ω',
Supp(ρ') is cpt and in Ω.

$$\Rightarrow \int_{\Omega'} (\Delta\chi)\cdot\phi = \int_{\Omega'} \chi\cdot\rho' \quad \forall \chi \in \mathcal{C}_c^\infty(\Omega')$$

<u>Fact</u>: Let K be the kernel fcn: $K(x) := \frac{1}{2\pi}\log|x|$.

(not defined at 0, but is locally integrable fcn. on $\mathbb{R}^2$).

- $(K * -)$ is the "inverse" of $\Delta(-)$:
  - If $\sigma \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, then $K * (\Delta\sigma) = \sigma$.
  - If $f \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, then $\Delta(\underbrace{K * f}_{\text{also smooth.}}) = f$.

where $(K * f)(x) = \int_{\mathbb{R}^2} K(y) f(x-y)\, d\mu_y$ is the <u>convolution</u>.

$$\Rightarrow \exists\, \psi' \in \mathcal{C}^\infty \text{ st. } \Delta\psi' = \rho' \qquad (\psi' = K * \rho')$$

$$\Rightarrow \int_{\Omega'} (\Delta\chi)\phi = \int_{\Omega'} \chi\cdot\Delta\psi' = \int_{\Omega'} (\Delta\chi)\cdot\psi' \quad \forall \chi \in \mathcal{C}_c^\infty$$

It remains to show: If $\int_{\Omega'} (\Delta \chi)\cdot \phi = 0 \quad \forall \chi \in \mathcal{C}_c^\infty$,

then ϕ is smooth.

(i.e. ϕ is a "weak sol$^{\underline{n}}$" of the heat eq$^{\underline{n}}$ $\Delta\phi = 0$).

- Using the mean-value property of harmonic fcns, one can construct an explicit sm. fcn. $B \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ supp. on $|z| \leq \varepsilon$. s.t.

  " Suppose $\psi \in \mathcal{C}^\infty(\mathbb{R}^2)$ and $\mathrm{Supp}(\Delta\psi) \subseteq \underset{\text{cpt}}{J} \subseteq \mathbb{C}$.

  Then $B * \psi - \psi$ is supported in the ε-nbhd of J."

- Observe that if ϕ is a sm. sol$^{\underline{n}}$ of $\Delta\phi = 0$, then we have

  $$B * \phi = \phi \quad \text{in } \Omega'.$$

- $\forall$ L^2-fcn ϕ, $B * \phi \in \mathcal{C}^\infty$.

  Therefore, proving $\phi \in \mathcal{C}^\infty \iff$ proving $B * \phi = \phi$ in Ω'.

- It suffices to show: $\forall \chi \in \mathcal{C}_c^\infty(\Omega')$,

  $$\int_{\Omega'} \chi \cdot (B * \phi - \phi) = 0.$$

- $K * \chi$ is smooth, with $\Delta(\underbrace{K * \chi}) = \chi, \to$ supp. in Ω'.

  $\Rightarrow \underbrace{B * K * \chi - K * \chi}_{h}$ supp. in Ω (containing ε-nbhd of Ω')

$\Delta h = B * \chi - \chi.$

• $0 = \int_\Omega (\Delta h)\phi = \int_\Omega (B * \chi - \chi)\phi$

$$= \int_\Omega \chi (B*\phi - \phi). \quad \square$$

Rmk: (on harmonic forms).

M: cpt. Riemannian mfld $\rightsquigarrow$ inner product on $A^k(M)$

Heuristic: In a class $[\alpha] \in H^k_{dR}(M)$, the infimum of $\|\cdot\|$ of a repn of $[\alpha]$ is achieved by harmonic form. $(\Delta\alpha = 0)$

Suppose α achieves the min. of $\|\alpha + d\beta\|$.

Then $\|\alpha + t d\beta\|^2 \geq \|\alpha\|^2$ holds for all $t \in \mathbb{R}$, $\beta \in A^{k-1}$

$\|\alpha + t d\beta\|^2 = \|\alpha\|^2 + 2t\langle \alpha, d\beta\rangle + t^2\langle d\beta, d\beta\rangle$.

$\Rightarrow \langle \alpha, d\beta\rangle = 0$, $\langle \alpha, d\beta\rangle = \langle d^*\alpha, \beta\rangle$ $\Rightarrow d^*\alpha = 0$.

Note: $\Delta\alpha = 0 \Leftrightarrow d\alpha = d^*\alpha = 0$.

Rmk: • We started with a class $[\alpha] \in H^k_{dR}(M)$.

The auxillary structure, (in this case, the metric on M) $\rightsquigarrow \|\cdot\|$.

$\rightsquigarrow$ unique (harmonic) representative of the class $[\alpha]$.

• Simpler example: $W \subseteq V$ subspace of vector space.

Consider a class $[\alpha] \in V/W$.

A priori, there is no preferred choice of representative.

An auxillary structure, an inner product on V $\rightsquigarrow$ $S(v) = \|v\|$

$\rightsquigarrow$ unique representative of $[\alpha]$, (its orthogonal proj. onto $W^{\perp}$).

- Donaldson–Uhlenbeck–Yau:

  Consider the space of connections on a holo. v.b. $E \to X$.

  The auxillary structure. (a Kähler metric on X) gives a preferred choice of connection (Hermitian-Yang-Mills connection), when E is polystable.

- Thomas-Yau Conjecture): (sLag repn.)

Slogan: "metric" $\rightsquigarrow$ preferred representative in each class.

(harmonic form, orthogonal projection, $\cdots$)