Last time: X - cpt. R.S. ω: mero. 1-form.

Then $\displaystyle\sum_{\substack{p:\text{pole}\\ \text{of } \omega}} \mathrm{res}_p(\omega) = 0.$

Coro: $f: X \to \mathbb{C}$ mero. fcn. on a cpt. R.S.

Then $\displaystyle\sum_{\substack{p:\text{ zeros}\\ \text{or poles of } f}} \mathrm{ord}_p(f) = 0.$ (# zeros = # poles on cpt. R.S.)

($\omega = \frac{df}{f}$ is a mero. 1-form with poles at the zeros & poles of f, with $\mathrm{res}_p(\omega) = \mathrm{ord}_p(f)$.)

Def: ω - mero. 1-form. $\displaystyle \deg(\omega) := \sum_{\substack{p:\text{ zeros}\\ \text{or poles of } \omega}} \mathrm{ord}_p(\omega)$

(Note: The order of zero/pole of ω is well-defined, i.e. indep. of the choice of local coord.)

Fact: The degree is indep. of the choice of the mero. 1-form!
i.e. it depends only on the cpt. R.S. X.

(ω_1/ω_2 defines a mero. fcn. on X, w/ $\mathrm{ord}_p(\omega_1/\omega_2) = \mathrm{ord}_p(\omega_1) - \mathrm{ord}_p(\omega_2)$).

e.g. $\mathbb{P}^1$, $\omega = dz$, has a double pole at ∞.

$\Rightarrow \deg(\omega) = -2.$

<u>Fact</u>: X - cpt. R.S. of genus g, then $\deg(\omega) = 2g-2$ $\forall \omega$ - mero. 1-form.

<u>pf</u>: Choose any mero. fcn. $f: X \longrightarrow \mathbb{C}$. (nonconst.)

[We'll show later that f always exists, e.g. by Riemann-Roch:

$$h^0(D) - h^0(K-D) = \deg D - g + 1.$$

$\Rightarrow \forall \{p_1, \dots, p_{g+1}\} \subseteq X$, $\exists$ nonconst. mero fcn. on X w/ at worst simple poles at p_i, and no other poles.]

[We'll see also a concrete construction of such mero. fcn.]

Equivalent, we have nonconst. holo fcn $f: X \longrightarrow \mathbb{CP}^1$.

- up to composing w/ $\mathrm{Aut}(\mathbb{CP}^1)$, we may assume ∞ is not a crit. value
- Consider the mero. 1-form $\alpha := f^*(dz)$.
- There are (counted w/ multiplicity) $\underline{2\deg(f)}$ poles in X over ∞.
- α has no zeros / poles at a non-critical pt.
- At a crit. pt. p, f is locally like $w \longmapsto w^k$, $k = \mathrm{mult}_p f$), and α is locally $d(w^k) = k w^{k-1} dw$ has zero of order $k-1$.
- Then,

$$\begin{aligned}
\deg(\omega) &= -2\deg(f) + \sum_{p \in \mathrm{Crit}(f)} (\mathrm{mult}_p(f) - 1) \\
&= -\chi(\mathbb{P}^1) \cdot \deg(f) + \sum_p (\mathrm{mult}_p(f) - 1) \\
&= -\chi(X) \text{ by Riemann-Hurwitz} \\
&= 2g-2. \quad \square
\end{aligned}$$

Rmk: The fact that $\sum \mathrm{res}_p(f) = 0$ for mero. fcn. f.

$\sum \mathrm{res}_p(\omega) = 2g-2$ for mero 1-form ω

is closely related to the fact that:

$\deg(\mathcal{O}_X) = 0$, where $\mathcal{O}_X$ is the trivial line bundle, or the sheaf of holo. fcns.

$\deg(\omega_X) = 2g-2$, where ω_X is the canonical line bundle, (holo. cotangent bundle). or the sheaf of holo. 1-forms.

and f, ω are mero. sections of these two line bundles.

Rmk: There are other ways to obtain $\deg(\omega) = 2g-2$ (e.g. topological), and one can prove Riemann-Hurwitz using this fact:

$f: X \longrightarrow Y$. $\deg(f) = d$

- Choose a mero. 1-form ω on Y. (we'll see later that $\dim_{\mathbb{C}}\{\text{holo. 1-form}\} = g$, for $\mathbb{P}^1$, just take dz).

  s.t. the poles are not critical values.

- at regular values. $f^*(\omega)$ has d times of the order of ω.
- at critical value, locally $w \mapsto w^k$, so $f^*(\omega)$ locally is: $\omega = g(z^k)\, d(z^k) = k z^{k-1} g(z^k)\, dz$

$$\Rightarrow \underbrace{\deg(f^*(\omega))}_{= -\chi(X)} = d \cdot \overbrace{\deg(\omega)}^{= -\chi(Y)} + \sum_{p:\, \mathrm{Crit}(f)} (\mathrm{mult}(p) - 1) \quad \square$$

if g has order ℓ. then $f^*\omega$ has order $k-1+k\ell$.

sum over same fibe $\xrightarrow{\Sigma}$ $d\ell$

Def: The <u>Dolbeault cohomology:</u>

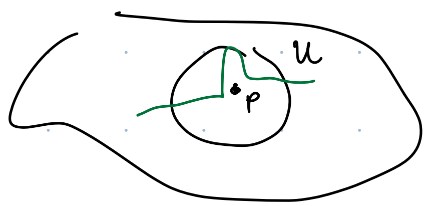
$$\mathcal{H}^{p,q}(X) := \frac{\ker\left(A^{p,q}(X) \xrightarrow{\bar\partial} A^{p,q+1}(X)\right)}{\operatorname{Im}\left(A^{p,q-1}(X) \xrightarrow{\bar\partial} A^{p,q}(X)\right)}$$

<u>For R.S.:</u>

- $\mathcal{H}^{0,0}(X) = \ker\left(A^0(X)_{\mathbb{C}} \xrightarrow{\bar\partial} A^{0,1}(X)\right) = \{\text{holo. fcns}\}.$
- $\mathcal{H}^{1,0}(X) = \ker\left(A^{1,0}(X) \xrightarrow{\bar\partial} A^2(X)_{\mathbb{C}}\right) = \{\text{holo. 1-forms}\}.$
- $\mathcal{H}^{0,1}(X) = \dfrac{A^{0,1}(X)}{\operatorname{Im}\left(A^0(X)_{\mathbb{C}} \xrightarrow{\bar\partial} A^{0,1}(X)\right)}$
- $\mathcal{H}^{1,1}(X) = \dfrac{A^2(X)_{\mathbb{C}}}{\operatorname{Im}\left(A^{1,0}(X) \xrightarrow{\bar\partial} A^2(X)_{\mathbb{C}}\right)}$

<u>Rmk:</u> $\mathcal{H}^{0,1}(X)$ naturally arise when we attempt to construct mero. fcns.

- say we want to construct a mero. fcn on X with a simple pole at p, and no other poles, (we know the answer is $X \cong \mathbb{P}^1$, since $X \to \mathbb{P}^1$ is of deg 1).
- Let z be a local coord. of p. Then $1/z$ is a mero. fcn around p.



choose a cut-off fcn β:
- smooth fcn, supp in $\mathcal{U}$,
- equal to 1 near p.

Then $\beta \cdot \frac{1}{z}$ is a smooth fcn on $X \setminus \{p\}$.

- Finding a mero. fcn w/ a pole at p $\Leftrightarrow$ finding sm. fcn. g on X s.t. $g + \beta \cdot \frac{1}{z}$ is holo. on $X \setminus \{p\}$.
- $A := \bar\partial\left(\beta \cdot \frac{1}{z}\right) = \underbrace{(\bar\partial \beta)}_{0 \text{ near } p} \cdot \frac{1}{z}$ is a $(0,1)$-form on the whole X! (extending by 0 over p)

- So the problem is equivalent to solving $g \in A^0(X)_{\mathbb{C}}$

$$\bar\partial g = -A$$

for the given $A \in A^{0,1}(X)$. $\left(A = \bar\partial(\beta \cdot \tfrac{1}{z})\right)$

- A sol^n exists $\iff$ $[A] = 0$ in $H^{0,1}(X) = \dfrac{A^{0,1}(X)}{\mathrm{Im}(A^0(X)_{\mathbb{C}} \to A^{0,1}(X))}$.

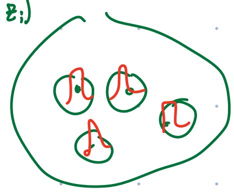
Note that even if such g doesn't exist, the class $[A]$ is (up to scaling) a well-defined class in $H^{0,1}(X)$ associated to the point p!

- Suppose ϕ is another sm. fcn. on $X \setminus \{p\}$ which restricts to a mero. fcn. w/ a pole at p, then, $\phi - \lambda \beta \cdot \frac{1}{z}$ extends to a sm. fcn. on X.

$$\Rightarrow \quad [\bar\partial \phi] = \lambda \cdot [A] \in H^{0,1}(X).$$

Similarly, we ask if there exists a mero. fcn. w/ poles at a subset of $\{p_1, \dots, p_n\}$.

Using the same argument, each $p_i \rightsquigarrow [A_i] \in H^{0,1}(X)$. $\bar\partial(\beta_i \cdot \frac{1}{z_i})$



Such mero. fcn exists iff. $\exists \lambda_1, \dots, \lambda_n$ not all 0

st. $\lambda_1 [A_1] + \cdots + \lambda_n [A_n] = 0$ in $H^{0,1}(X)$.

<u>Coro</u>: Suppose $\dim H^{0,1}(X) = n$. Then, given $p_1, \dots, p_{n+1} \in X$, $\exists$ mero. fcn. on X w/ simple poles at some subset of $\{p_1, \dots, p_{n+1}\}$.

(Later, we'll see that $\dim H^{0,1}(X) = g = g(X)$.)

More generally, we can ask, given $p_1, \dots, p_n$ and for each p_i,

$$P_i(z_i) = \frac{a_{-k_i}}{z_i^{k_i}} + \dots + \frac{a_{-1}}{z_i} \quad \text{in local coord. } z_i \text{ near } p_i.$$

Does there exist a mero. fcn. on X s.t. it's holo. in $X \setminus \{p_1, \dots, p_n\}$, with the prescribed principle part?

This can be formulated in terms of sheave:

$$0 \longrightarrow \underset{\substack{\uparrow \\ \text{sheaf of} \\ \text{holo. fcns.}}}{\mathcal{O}_X} \longrightarrow \underset{\substack{\uparrow \\ \text{sheaf of} \\ \text{mero. fcns}}}{\mathcal{M}_X} \longrightarrow \mathcal{M}_X / \mathcal{O}_X \longrightarrow 0.$$

- The prescribed data is a global section $H^0(X, \mathcal{M}_X/\mathcal{O}_X)$. mero. fcn. on X
- The problem is, given $\alpha \in H^0(X, \mathcal{M}_X/\mathcal{O}_X)$, does there exist $\beta \in H^0(\mathcal{M}_X)$ s.t. $\beta \longmapsto \alpha$ in $H^0(\mathcal{M}_X) \longrightarrow H^0(\mathcal{M}_X/\mathcal{O}_X)$.
- The <u>obstruction</u> is precisely given by:

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{M}_X) \to H^0(X, \mathcal{M}_X/\mathcal{O}_X) \to \underbrace{H^1(X, \mathcal{O}_X)}_{\substack{\cong \\ H^{0,1}(X)}} \to \cdots$$

Therefore, if a class $\alpha \in H^0(X, \mathcal{M}_X/\mathcal{O}_X) \longmapsto 0 \in H^{0,1}(X)$, then such β exists.

Let's compare $\mathcal{H}^{1,0}(X) = \{\text{holo. 1-forms on } X\}$

and $\mathcal{H}^{0,1}(X) = \dfrac{A^{0,1}(X)}{\mathrm{Im}\,(A^0(X)_{\mathbb{C}} \xrightarrow{\bar\partial} A^{0,1}(X))}$

Consider the map

$$\begin{tikzcd}
\mathcal{H}^{1,0}(X) \arrow[r] & A^{0,1}(X) \arrow[dr] & \\
\alpha \arrow[r, mapsto] & \bar\alpha \arrow[dr, mapsto] & \mathcal{H}^{0,1}(X) \\
& & {[\bar\alpha]}
\end{tikzcd}$$

(locally, $\alpha = f(z)\,dz$, $\bar\alpha = \overline{f(z)}\,d\bar z$).

<u>Claim</u>: $\mathcal{H}^{1,0}(X) \longrightarrow \mathcal{H}^{0,1}(X)$ is an isom.

$\alpha \longmapsto [\bar\alpha]$

* <u>Injective</u>. if $[\bar\alpha] = 0$, i.e. $\bar\alpha = \bar\partial f$ for some smooth fcn $f \in A^0(X)_{\mathbb{C}}$

$\Rightarrow \alpha = \partial \bar f$

Since $\alpha \in \mathcal{H}^{1,0}(X) \Rightarrow \bar\partial \alpha = \bar\partial \partial f = 0$.

<u>Def</u>: A fcn $u: X \to \mathbb{C}$ is <u>harmonic</u> if $\Delta u = 0$, where Δ is the Laplace operator: $\Delta = 2i\bar\partial\partial : A^0(X)_{\mathbb{C}} \to A^2(X)_{\mathbb{C}}$.

Explicitly, $\Delta u = 2i\,\frac{1}{4}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) u \, dz\, d\bar z$

$= -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx\, dy$

$\Leftrightarrow$ ∂u is a holo. 1-form.

$\Leftrightarrow$ $\bar\partial u$ is an anti-holo. 1-form (i.e. $\partial(*) = 0$)

$\Leftrightarrow$ locally, $u = f + \bar g$, where f, g holo.

On cpt. R.S, the only harmonic fcns $\Delta f = 0$ are constant fcns:

1) Maximal principle of harmonic fcn. ($\Delta(u+iv) = 0 \Rightarrow \Delta u = \Delta v = 0$.) $\square$

2) Dirichlet integral:

Let $\alpha \in A^{1,0}(X)$. consider the 2-form: $i \cdot \alpha \wedge \bar{\alpha}$. ($\alpha = f\,dz$ locally).

Locally, $i\alpha \wedge \bar{\alpha} = i \cdot |p|^2 dz \wedge d\bar{z} = 2|p|^2 dx dy$.

So, $i\alpha \wedge \bar{\alpha}$ is a positive 2-form.

$$\|\alpha\|^2 := \int_X i\alpha \wedge \bar{\alpha} \geq 0.$$

Integration by parts + Stokes' thm to eliminate boundary term. $\Rightarrow \|df\|^2 = \int_X f \Delta f = 0$ for harmonic fcn.

$\Rightarrow df = 0. \Rightarrow f \equiv$ const. for harmonic fcns on cpt. R.S.

This concludes the proof that

$$\begin{aligned} H^{1,0}(X) &\longrightarrow H^{0,1}(X) \quad \text{is injective.} \\ \alpha &\longmapsto [\bar{\alpha}] \end{aligned}$$

* <u>Surjective</u>: $\forall\ [\theta] \in H^{0,1}(X) = \dfrac{A^{0,1}(X)}{\mathrm{Im}(A^0(X)_{\mathbb{C}} \xrightarrow{\bar{\partial}} A^{0,1}(X))}$

want to find holo. 1-form α st. $[\bar{\alpha}] = [\theta]$,

i.e. find α & $f \in A^0(X)_{\mathbb{C}}$ st. $\bar{\alpha} - \theta = \bar{\partial} f$.

$\Leftrightarrow$ Find $f \in A^0(X)_{\mathbb{C}}$ st. $\overline{\theta + \bar{\partial} f}$ is holo. 1-form.

i.e. $\bar{\partial}\left(\overline{\theta + \bar{\partial} f}\right) = \partial(\theta + \bar{\partial} f) = 0$.

$\Leftrightarrow \exists f \in A^0(X)_{\mathbb{C}}$ st $\partial\bar{\partial} f = -\partial\theta = -d\theta$

Thm: X - cpt. R.S. $\rho \in A^2(X)_{\mathbb{C}}$.

" $\exists f \in A^0(X)_{\mathbb{C}}$ st. $\Delta f = \rho$ " $\Longleftrightarrow$ " $\int_X \rho = 0$ ".

Moreover, the sol^{n} is unique up to constant.

Rmk: The uniqueness follows directly from $\Delta f = 0 \Rightarrow f = \text{const}$.

In our situation, we have " $\partial\bar{\partial} f = -d\theta$ "

By Stokes' thm, $\int_X d\theta = 0$.

Therefore, such f exists by the above thm.

This completes the proof of $H^{1,0}(X) \cong H^{0,1}(X)$. $\square$

Claim: The bilinear map
$$B: H^{1,0}(X) \times H^{0,1}(X) \longrightarrow \mathbb{C}.$$
$$(\alpha, [\theta]) \longmapsto \int_X \alpha \wedge \theta.$$

is <u>non-degenerate</u>, therefore induces $H^{0,1}(X) \cong H^{1,0}(X)^*$.

* <u>well-defined</u>: $\int_X \alpha \wedge (\theta + \bar{\partial} f) = \int_X \alpha \wedge \theta + \int_X \alpha \wedge \bar{\partial} f$ $\quad -\int_X \bar{\partial}(f\alpha) = -\int_X d(f\alpha) = 0.$

$$\alpha \wedge \bar{\partial} f = -\bar{\partial} f \wedge \alpha \underset{\bar{\partial}\alpha = 0}{=} -\bar{\partial}(f\alpha)$$

* <u>non-deg.</u>: Easy to check $B(\alpha, \bar{\alpha}) \neq 0 \quad \forall \alpha \in H^{1,0}(X) \setminus \{0\}$.

(Locally, $\alpha = f\,dz$, $\alpha \wedge \bar{\alpha} = |f(z)|^2 dz\, d\bar{z} = -2i\, |f(z)|^2 dx\, dy$.) $\square$

Claim: $\mathcal{H}^{1,0}(X) \oplus \mathcal{H}^{0,1}(X) \cong \mathcal{H}^1_{dR}(X)_{\mathbb{C}} = \dfrac{\ker(A^1(X)_{\mathbb{C}} \xrightarrow{d} A^2(X)_{\mathbb{C}})}{\operatorname{Im}(A^0(X)_{\mathbb{C}} \xrightarrow{d} A^1(X)_{\mathbb{C}})}$

$\alpha \longmapsto [\alpha]$

α d-closed.

$\Rightarrow \alpha \in \ker(A^1 \xrightarrow{d} A^2)$

$\dfrac{A^{0,1}}{\operatorname{Im}(A^0 \xrightarrow{\bar\partial} A^{0,1})} = H^{0,1}(X) \ni [\beta]$

$\beta \in A^{0,1}$ may not be d-closed.

$H^{0,1}(X) \to H^{1,0}(X)$, $[\beta] \mapsto \overline{\beta} \leftarrow \overline{\beta + \bar\partial f}$

$H^{1,0}(X) \longrightarrow [\overline{\beta}]$

More explicitly, $\exists f \in A^0$ s.t. $\overline{\beta + \bar\partial f}$ is holo. 1-form, thus d-closed. $[\beta] = [\beta + \bar\partial f]$

* Injective: Suppose $\alpha \oplus [\beta] \longmapsto 0 \in \mathcal{H}^1_{dR}$.

α : holo. 1-form

$\beta \in A^{0,1}$. $f \in A^0$, $\overline{\beta + \bar\partial f}$: holo 1-form.

$$\alpha + \beta + \bar\partial f \in \operatorname{Im}(A^0 \xrightarrow{d} A^1)$$

i.e. $\exists g \in A^0$ st. $dg = \underset{(1,0)}{\alpha} + \underset{(0,1)}{\beta + \bar\partial f}$

$dg = \partial g + \bar\partial g$

$\Rightarrow \partial g = \alpha \longrightarrow \bar\partial \partial g = \bar\partial \alpha = 0 \Rightarrow g$ harmonic $\Rightarrow$ const.

$\Rightarrow \alpha = \partial g = 0$

$\bar\partial g = \beta + \bar\partial f$

$\Downarrow$

$\partial \bar g = \overline{\beta + \bar\partial f}$ hol.

same proof $\Rightarrow \beta + \bar\partial f = 0$.

* Surjective: $\forall\, \omega \in A^1$, $d\omega = 0$.

want: $\exists\, \alpha$: holo. 1-form, β: anti-holo. 1-fm, $g \in A^0$ st. $\alpha + \beta = \omega + dg$

$\iff \exists\, g \in A^0$ st $\bar{\partial}(\omega^{1,0} + \partial g) = 0$ and $\partial(\omega^{0,1} + \bar{\partial} g) = 0$.

such g exists by the main Thm, and $d\omega = 0$.

Coro: $\dim H^{1,0}(X) = \dim H^{0,1}(X) = g = g(X)$, since $\dim H^1(X) = 2g$.

i.e. $\dim_{\mathbb{C}} \{\text{holo. 1-forms}\} = g$.

Claim: $H^{1,1}(X) \cong H^2_{dR}(X)_{\mathbb{C}} = \dfrac{A^2(X)_{\mathbb{C}}}{\operatorname{Im}(A^1(X)_{\mathbb{C}} \xrightarrow{d} A^2(X)_{\mathbb{C}})}$ $\left(\cong H^2(X, \mathbb{C}) \cong \mathbb{C}\right)$

$\|$

$\dfrac{A^2(X)_{\mathbb{C}}}{\operatorname{Im}(A^{1,0}(X) \xrightarrow{\bar{\partial}} A^2(X)_{\mathbb{C}})}$

i.e. $\operatorname{Im}(A^{1,0} \xrightarrow{\bar{\partial}} A^2) \overset{?}{=} \operatorname{Im}(A^1 \xrightarrow{d} A^2)$

$\uparrow$

again, follows from the Main Thm.