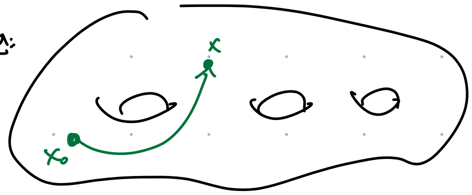
Rmk: It'll be important to understand <u>holo. 1-forms</u> (or more generally, <u>harmonic 1-forms</u>), on compact R.S. (<u>Note</u>: There is no nontrivial holo. <u>fcn</u>. on cpt. R.S., but there are plenty of holo. <u>1-forms</u>).

Locally, it's given by $f(z)\,dz$, where f is holo.

<u>e.g.</u> X - cpt. R.S.

Suppose there is a nowhere-vanishing holo. 1-form on X, then $X \cong \mathbb{C}/\Lambda$.

<u>Idea:</u>



Let ω be a nowhere vanishing holo. 1-form.

We'd like to consider holo. fcn "$\int_{x_0}^{x} \omega$"

But it's not well-defined, since it depends on the homotopy class of path. So:

$$\begin{tikzcd} \tilde{X} \arrow[d, "p"'] \\ X \end{tikzcd}$$

universal cover

Now we can define $F(\tilde{x}) := \int_{\tilde{x}_0}^{\tilde{x}} p^*\omega$. $\rightarrow$ holo fcn <u>$F: \tilde{X} \to \mathbb{C}$</u>

fixed basept. $\rightarrow \tilde{x}_0$

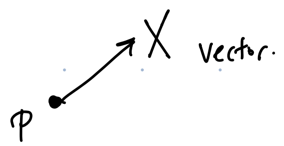
- Since ω has no zero, F is a local homeomorphism. (In fact, a covering map)
- Since $\mathbb{C}$ is simply connected, $F: \tilde{X} \cong \mathbb{C}$.
- $X \cong \mathbb{C}/\{\text{holo. auto}\}$
- By classification of $\mathrm{Aut}(\mathbb{C})$, we have $X \cong \mathbb{C}/\Lambda$

§ Tangent space

Note: It's not obvious how to generalized the notion of "tangent vectors" to manifolds. In fact, it's a challenging problem for topological mfd. (i.e. transition maps are homeom., without other assumptions).

We'll give 2 definitions of tangent spaces for smooth manifolds (i.e. transition maps are C^∞- functions)

e.g. In $\mathbb{R}^n$:



For a smooth fcn f near p, we have the directional derivative

$$Xf := (D_X f)(p) := \lim_{t\to 0} \frac{f(p+tX)-f(p)}{t}.$$

It's a "derivation", i.e.

1) linear: $X(af+bg) = aX(f) + bX(g)$
2) Lebnitz rule: $X(fg) = Xf \cdot g(p) + f(p) \cdot Xg.$

Conversely, a derivation $(f \mapsto Xf)$ determines the vector X.

Def: For a smooth mfld M, and $p \in M$, Define:

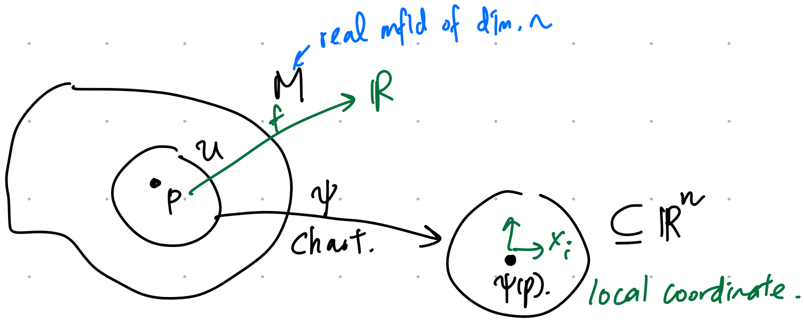
$C_p^\infty :=$ space of germs of C^∞ functions at p.

$$= \left\{ \begin{array}{c} C^\infty \text{ fcns defined on} \\ \text{a nbhd. of } p \end{array} \right\} \Big/ \begin{array}{l} f \sim g \text{ if } f|_U = g|_U \\ \text{for some } p \in U. \end{array}$$

Def: The tangent space

$$T_pM := \{ \text{derivations } C_p^\infty \longrightarrow \mathbb{R} \}.$$

e.g.



" $\left.\frac{\partial}{\partial x_i}\right|_p$ " are examples of tangent vectors:

$$\left.\frac{\partial}{\partial x_i}\right|_p f := \frac{\partial (f \circ \psi^{-1})}{\partial x_i}(\psi(p)).$$

Ex:

- $\forall X \in T_pM$, $X(\text{const. fcn.}) = 0$.
- In local coord., $f(x_1, \dots, x_n) = f(0, \dots, 0) + \sum x_i g_i(x_1, \dots, x_n)$ where $g_i \in C_0^\infty$ with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$. (use fund. thm. of calculus).
- $\forall X \in T_pM$, we have $X = \sum_{i=1}^n X(x_i) \left.\frac{\partial}{\partial x_i}\right|_p$.

  Therefore, $\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p \right\}$ form a basis of T_pM.

Prop: $T_pM \cong (m_p / m_p^2)^*$, where:

- $m_p = \{ f \in C_p^\infty \mid f(p) = 0 \}$ is the maximal ideal of C_p^∞.
- $m_p^2 = \{ \text{finite sums } \sum f_i g_i \text{ with } f_i, g_i \in m_p \}$.

pf: $\forall X \in T_pM$, we have $X: m_p \longrightarrow \mathbb{R}$.

For $f_i, g_i \in m_p$:

$$X\left(\sum f_i g_i\right) = \sum X(f_i) \cdot \underbrace{g_i(p)}_{0} + \sum \underbrace{f_i(p)}_{0} \cdot X(g_i) = 0.$$

$\Longrightarrow$ descends to $m_p / m_p^2 \longrightarrow \mathbb{R}$.

Conversely, given $\alpha: m_p/m_p^2 \longrightarrow \mathbb{R}$. we define:

$$X_\alpha f := \alpha(f - f(p))$$ and verify that it's a derivation.

$$\begin{aligned} X_\alpha(fg) &= \alpha(fg - f(p)g(p)) \\ &= \alpha(\underbrace{(f-f(p))(g-g(p))}_{\text{in } m_p^2} + (f-f(p))g(p) + f(p)(g-g(p))) \\ &= X_\alpha f \cdot g(p) + f(p) \cdot X_\alpha g. \quad \square \end{aligned}$$

§ Tangent maps

Let $f: M \longrightarrow N$ be a smooth map btw smooth mflds.

$$\begin{array}{c} M \\ \Downarrow \\ p \end{array}$$

Tangent map: $df_p : T_pM \longrightarrow T_{f(p)}N$:

$$X \longmapsto df_p X \qquad h \in C^\infty_{f(p)}.$$

$$(df_p X)(h) := X(h \circ f)$$

Rmk: It's the generalization of derivative of a map in calculus, and is the linearization of the map f at $p \in M$.

Rmk: It satisfies the chain rule: $M \xrightarrow{f} N \xrightarrow{g} S$

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

Note: The proof in Calculus is a bit tricky, but the proof now is completely formal:

$$(d(g\circ f)_p X)(h) = X(h \circ g \circ f) = (df_p X)(h \circ g)$$
$$= (dg_{f(p)}(df_p X))(h).$$

Rmk: Each local coord. x_i is a C^∞ fcn at p, and

$$dx_{i,p}\left(\frac{\partial}{\partial x_j}\Big|_p\right) = \frac{\partial x_i}{\partial x_j} = \delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

$\Rightarrow$ the "differentials" $\{dx_1, \ldots, dx_n\}$ form a dual basis of the cotangent space $T_p^*M := (T_pM)^* = \mathrm{Hom}(T_pM, \mathbb{R})$. w.r.t. $\left\{\frac{\partial}{\partial x_i}\right\}$.

Moreover, $$df = \sum_{i=1}^{n} \frac{\partial (f \circ \varphi^{-1})}{\partial x_i} dx_i .$$

§ Tensor Algebra:

Recall: Given vector spaces $V = \langle v_1, \dots, v_n \rangle$, $W = \langle w_1, \dots, w_m \rangle$.

One can define their tensor product $V \otimes W = \langle v_i \otimes w_j \rangle$ (of dim. $n \cdot m$)

and the definition is indep. of the choice of basis.

Def: The (r,s)-tensors of a vector space V:

$$T^{r,s}(V) := \underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s} \cong \underset{\text{multilinear fcns.}}{\mathrm{Hom}(V^{\otimes s}, V^{\otimes r})}$$

e.g.

- $T^{1,2}(V) = V \otimes V^* \otimes V^* \cong \mathrm{Hom}(V^{\otimes 2}, V)$

  $$T = \sum_{i,j,k} a_{ijk} v_i \otimes v_j^* \otimes v_k^* \longleftrightarrow T(v_j, v_k) = \sum a_{ijk} v_i .$$

- $T^{0,2}(V) = V^* \otimes V^*$

  e.g. metric tensor $g = \sum g_{ij} dx_i \otimes dx_j$.

- $T^{1,3}(V) = V \otimes (V^*)^{\otimes 3}$

  e.g. curvature tensor $R = \sum R^i_{jkl} \frac{\partial}{\partial x_i} \otimes dx^j \otimes dx^k \otimes dx^l$.

In our case, $V = T_pM$, $V^* = T_p^*M$.

On a local chart $(\mathcal{U}, \varphi, x_1, \dots, x_n)$, the tangent bundle $T\mathcal{U}$ has a framing $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$, and the cotangent bundle has a dual framing $\{ dx_1, \dots, dx_n \}$

Def: A tensor field $T \in T^{r,s}(\mathcal{U})$ (it actually means $T^{r,s}(T\mathcal{U})$) is smooth if

$$T = \sum \underbrace{T^{i_1,\dots,i_r}_{j_1,\dots,j_s}}_{\substack{\uparrow \\ \text{is smooth fcn in } \mathcal{U}.}} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}$$

Rmk: For a smooth map $f: M \longrightarrow N$, we can induce

$$T_pM \xrightarrow{f_*} T_{f(p)}N, \quad \text{and} \quad T^*_{f(p)}N \xrightarrow{f^*} T_pM$$

$$\omega \longmapsto (f^*\omega)v := \omega(f_* v)$$

$\leadsto$ For $T = \sum h_{j_1,\dots,j_s}\, dy_{j_1} \otimes \cdots \otimes dy_{j_s} \in T^{0,s}(V)$, $f(p) \in V$

we have $$f^*T = \sum (h_{j_1,\dots,j_s} \circ f)\, d(y_{j_1} \circ f) \otimes \cdots \otimes d(y_{j_s} \circ f).$$

§ Exterior Algebra

- $\otimes V := \bigoplus_{r=0}^{\infty} (V^{\otimes r})$. tensor algebra.
- $I(V) := \langle \cdots \otimes a \otimes \cdots \otimes a \otimes \cdots \rangle \overset{\text{ideal}}{\subseteq} \otimes V$.
- $\Lambda(V) := \otimes V / I(V)$ exterior algebra

$$= \bigoplus_{r=0}^{\infty} \Lambda^r(V) \quad \text{where} \quad \Lambda^r(V) = \mathbb{R}\langle e_{i_1} \wedge \cdots \wedge e_{i_r} \mid i_1 < \cdots < i_r \rangle$$
$$\cong \mathbb{R}^{\binom{n}{r}}. \quad (n = \dim V)$$

We'll be focusing on alternating tensors on mflds:

$$A^r(M) := C^\infty(\Lambda^r(T^*M)) = \{\text{smooth alternating } r\text{-forms}\}.$$
(i.e. "$dx \wedge dy = -dy \wedge dx$")

e.g. $A^0(M) = C^\infty(M)$

$A^1(M) = \{\text{smooth 1-forms on } M\}$. ← locally: $f_1(x_1, \ldots, x_n)dx_1 + \cdots + f_n(x_1, \ldots, x_n)dx_n$

$A^2(M) = \{\text{smooth alternating 2-forms on } M\}$ ← locally: $\sum_{i<j} f_{ij}(x_1, \ldots, x_n)\, dx_i \wedge dx_j$

There is an important operator, Cartan differential operator d:

e.g. Let $f \in C^\infty(M)$, then df can be regarded as a 1-form,

locally: $df = \sum_i \frac{\partial(f \circ \psi)}{\partial x_i} dx_i$

e.g. Green's thm. $D \subseteq \mathbb{R}^2$.

$$\int_{\partial D} P\,dx + Q\,dy = \int_D (Q_x - P_y)\, dx \wedge dy.$$

Thm: $\exists!\ d: A^p(M) \longrightarrow A^{p+1}(M)$ such that:

1) $\forall f \in C^\infty(M)$, $df(X) = Xf$.
2) $d^2 = 0$
3) (Lebnitz rule) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$. for $\omega \in A^p(M)$.

Choose a local chart $(\mathcal{U}, \varphi)$ near a point $p \in M$.

Let $\omega = f_I \, dx_I = \sum f_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

Then $d\omega$ is given by:

$$d\omega = \sum df_{i_1 \cdots i_p} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

Recall that there is a pullback of $f: M \longrightarrow N$.

$$\begin{aligned} f^*: \ A^*(N) &\longrightarrow A^*(M) \\ \omega &\longmapsto f^*\omega \end{aligned}$$

where $(f^*\omega)(v_1, \dots, v_r) := \omega(f_* v_1, \dots, f_* v_r)$.,

and $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$. i.e. f^* is a ring homomorphism.

Moreover, $\underline{d f^*(\omega) = f^*(d\omega)}$. ($\underline{\underline{Ex}}$).

§ de Rham cohomology:

Consider: $$\cdots \longrightarrow A^{p-1}(M) \xrightarrow{d_{p-1}} A^p(M) \xrightarrow{d_p} A^{p+1}(M) \longrightarrow \cdots$$

It's a <u>complex</u>, i.e. $d^2 = 0$.

<u>Def</u>: The <u>p-th de Rham cohomology gp</u> of M

$$H^p_{dR}(M) := \frac{\ker d_p}{\operatorname{Im} d_{p-1}}.$$

$\ker d_p$ ← <u>closed</u> p-forms.

$\operatorname{Im} d_{p-1}$ ← <u>exact</u> p-forms.

One can also define the <u>compactly supported de Rham cohom.</u> $H^p_c(M)$ via the complex $$\cdots \longrightarrow A^{p-1}_c(M) \longrightarrow A^p_c(M) \longrightarrow A^{p+1}_c(M) \longrightarrow \cdots$$

where $A^p_c(M)$ are smooth alternating p-forms supp. on a cpt subset of M.

<u>Thm</u> (Poincaré lemma) If $\mathcal{U}$ is contractible, then $H^p_{dR}(\mathcal{U}) = 0$ $\forall p \geq 1$.

<u>Rmk</u>: If M is connected, then, for $f \in A^0(M) = C^\infty(M)$, $df = 0 \Leftrightarrow f \equiv \text{const}$.

$\Rightarrow$ $H^0_{dR}(M) \cong \mathbb{R}$.

Also, $H^0_c(M) \cong \begin{cases} \mathbb{R} & \text{if } M \text{ cpt.} \\ 0 & \text{if } M \text{ non-cpt.} \end{cases}$

<u>Fact</u>: Let M be a <u>cpt</u>. <u>oriented</u> mfd of dim. n, then $H^n_{dR}(M) \cong \mathbb{R}$.

<u>Rmk</u>: For $f: M \longrightarrow N$, since $d f^* = f^* d$, it induces

$$f^*: H^p_{dR}(N) \longrightarrow H^p_{dR}(M).$$

§ Integration on forms.

Def: M is orientable if $\exists$ atlas st. all transition maps have positive Jacobians, i.e. $\det\left(\frac{\partial x_i}{\partial y_j}\right) > 0$.

Fact: M^n is orientable $\iff$ $\exists\ \omega \in A^n(M)$ st. $\omega \neq 0$ everywhere on M.

pf: ($\Leftarrow$) On a local coord., write $\omega = f(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$.

Since $\omega \neq 0$ everywhere, $f > 0$ or $f < 0$ everywhere.

If $f < 0$, replace the chart by $(\mathcal{U}, (x_2, x_1, x_3, \dots, x_n))$.

Therefore, we can choose an atlas st. $\omega = f\, dx_1 \wedge \dots \wedge dx_n$ where $f > 0$ at each chart.

At overlap: $\omega = f(x_1, \dots x_n)\, dx_1 \wedge \dots \wedge dx_n$
$= g(y_1, \dots y_n)\, dy_1 \wedge \dots \wedge dy_n$ $\quad f, g > 0$.

$$f(x_1, \dots, x_n) = g(y_1, \dots y_n) \cdot \det\left(\frac{\partial x_i}{\partial y_j}\right)$$

$\Rightarrow$ $\det\left(\frac{\partial x_i}{\partial y_j}\right) > 0$. btw every charts.

($\Rightarrow$). Suppose M is oriented by an atlas $\{(\mathcal{U}, \phi_{\mathcal{U}})\}$.

Let $\omega_{\mathcal{U}} = \phi_{\mathcal{U}}^*(dx_1 \wedge \dots \wedge dx_n)$ $\quad \phi_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathbb{R}^n$

nonzero on $\mathcal{U}$.

<u>Fact</u>: (<u>Partition of unity</u>) $\exists\ \rho_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}$ C^∞.

st. $\sum \rho_{\mathcal{U}} = \mathbb{1}_M$.

$\omega := \sum \rho_{\mathcal{U}}\, \omega_{\mathcal{U}}$. is nonvanishing everywhere $\square$

<u>Def:</u> M · oriented with an atlas $\{\mathcal{U}, \phi\}$.

$\eta \in A_c^n(M)$.

$$\int_M \eta := \sum_{\mathcal{U}} \int_{\mathcal{U}} \rho_{\mathcal{U}} \eta = \sum_{\mathcal{U}} \int_{\mathbb{R}^n} (\phi_{\mathcal{U}}^{-1})^* (\rho_{\mathcal{U}} \eta).$$

<u>Note:</u> In topology, there is a notion of <u>singular homology gp.</u> $H_p(M;\mathbb{R})$ which is given by a complex:

$$\longrightarrow S_{p-1}(M) \xrightarrow{\partial_{p-1}} S_p(M) \xrightarrow{\partial_{p+1}} S_{p+1}(M)$$

$S_p(M)$ || free abel. gp. gen. by $\sigma : [0,1]^p \longrightarrow M$ conti. map.

M: cpt

$$H^p_{dR}(M) \xrightarrow[\cong]{\int} H_p(M;\mathbb{R})^* \cong H^p(M;\mathbb{R}).$$

$$[\omega] \longmapsto \int \omega : \left([\sigma] \longmapsto \int_\sigma \omega\right)$$

$$\int_\sigma \omega \stackrel{..}{=} \int_{[0,1]^p} \sigma^* \omega.$$

identifies the de Rham cohom. with the singular cohomology $/\mathbb{R}$.

§ Complex structure on R.S.

Let's specialize to X - R.S.

- Multiplication by $i = \sqrt{-1}$ defines a map $\mathbb{C} \xrightarrow{i} \mathbb{C}$.
- Its derivative $j : T_0\mathbb{C} \longrightarrow T_0\mathbb{C}$ is just i again under $T_0\mathbb{C} \cong \mathbb{C}$.

For a R.S. X, it has a complex str., which is a $\mathbb{R}$-linear map

$$J_p : T_pX \longrightarrow T_pX \quad \text{s.t.} \quad J_p^2 = -\mathrm{id}$$

and $\{J_p\}$ varies smoothly in $p \in X$.

↑

the derivative of the map $z \mapsto iz$ in local coordinates

- It induces $J_p^* : T_p^*X \longrightarrow T_p^*X$ on the cotangent space
- Consider $(T_p^*X) \otimes \mathbb{C} \cong \mathrm{Hom}(T_pX, \mathbb{C})$ complexified cotangent space.
- Since $(J_p^*)^2 = -\mathrm{id}$, $T_p^*X \otimes \mathbb{C}$ decomposes into i and $-i$ eigenspaces:

$$T_p^*X \otimes \mathbb{C} \cong \underset{\substack{\uparrow \\ +i}}{T_p^*X'} \oplus \underset{\substack{\uparrow \\ -i}}{T_p^*X''}$$

eigenspace

- The complex-valued 1-forms $A^1(X)_{\mathbb{C}} := A^1(X) \otimes C^\infty(X, \mathbb{C})$

(locally, $\alpha \in A^1(X)_{\mathbb{C}}$ looks like $f\,dx + g\,dy$, f, g: $\mathbb{C}$-valued sm. fcns)

$$\rightsquigarrow A^1(X)_{\mathbb{C}} \cong \underset{\substack{\uparrow \\ \text{valued in } T_p^*X'}}{A^{1,0}(X)} \oplus \underset{\substack{\uparrow \\ \text{valued in } T_p^*X''}}{A^{0,1}(X)}.$$

More explicitly, $\alpha \in A^1(X)_{\mathbb{C}}$ splits into $\alpha = \alpha^{1,0} + \alpha^{0,1}$,

where, if v is a tangent vector, then

$$\alpha^{1,0}(v) = \frac{1}{2}\left(\alpha(v) - i\alpha(Jv)\right)$$

$$\alpha^{0,1}(v) = \frac{1}{2}\left(\alpha(v) + i\alpha(Jv)\right)$$

holo. tangent space

Let $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$.

Then a $(1,0)$-form locally looks like $f(x,y)\,dz$.

$(0,1)$-form —— " —— $g(x,y)\,d\bar{z}$.

In particular, if f is a holo. fun. on X, then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= \frac{\partial f}{\partial z}dz + \underbrace{\frac{\partial f}{\partial \bar{z}}}_{=0}d\bar{z}$$

Since f holo.

$dz = dx + i\,dy$, $d\bar{z} = dx - i\,dy$

$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$, $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$

$\Rightarrow df \in A^{1,0}(X)$.

$\underset{\Omega}{\beta} = B\,dz$ locally $\quad d\beta = \frac{\partial B}{\partial \bar{z}}\,d\bar{z}\wedge dz = -2i\,\frac{\partial B}{\partial \bar{z}}\,dx\wedge dy$

-
$$\begin{tikzcd}
 & A^{1,0}(X) \arrow[dr, "\bar{\partial} = d|_{\Omega^{1,0}}"] & \\
A^0(X)_{\mathbb{C}} \arrow[ur, "\underset{\partial}{\overset{\frac{1}{2}(f_x - if_y)}{\parallel}}"] \arrow[dr, "\underset{\frac{1}{2}(f_x+if_y)}{\overset{\bar{\partial}}{\parallel}}"'] & & A^2(X)_{\mathbb{C}} \\
 & A^{0,1}(X) \arrow[ur, "\partial = d|_{\Omega^{0,1}}"'] &
\end{tikzcd}$$

$\Downarrow$

$\alpha = A\,d\bar{z}$ locally,

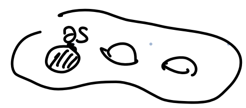
$d\alpha = \frac{\partial A}{\partial z}\,dz\wedge d\bar{z} = 2i\,\frac{\partial A}{\partial z}\,dx\wedge dy$.

Def: A <u>holo. 1-form</u> on X is an $\alpha \in A^{1,0}(X)$ with $\bar\partial \alpha = 0$.

(i.e. it's <u>closed</u>, and in $A^{1,0}$)

<u>e.g.</u>
- There is no holo. 1-forms on $\mathbb{P}^1$. (dz has a double pole at ∞: $d(\frac{1}{z}) = \frac{-1}{z^2}dz$)
- The set of holo 1-form on $\mathbb{C}/\Lambda$ is 1-dim'l ($\mathbb{C} \cdot dz$)
- Lots of holo. 1-form on $\mathbb{C}^\times$, e.g. $\frac{dz}{z}$.

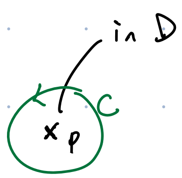
<u>Rmk:</u> Suppose S is a cpt. surface w/ boundary with a holo. 1-form α on it.



Then $\int_{\partial S} \alpha \underset{\substack{\uparrow \\ \text{Stokes}}}{=} \int_S d\alpha \underset{\substack{\uparrow \\ \alpha \text{ closed}}}{=} 0$

This is a version of Cauchy's thm for R.S.

<u>Rmk:</u> One can define <u>mero. 1-form</u> similarly. It's a holo. 1-form on $X \setminus D$, where $D \subseteq X$ discrete. s.t. locally it's $f(z)\,dz$ where f is mero. fcn.



$$\mathrm{Res}_p(\underset{\substack{\text{mero. 1-form}}}{\alpha}) := \frac{1}{2\pi i}\int_C \alpha .$$ <u>residue</u>

<u>Fact:</u> X - cpt. R.S. α - mero. 1-form on X.

Then $\sum_{p\text{-pole}} \mathrm{res}_p(\alpha) = 0$. (follows directly from Cauchy's Thm.)

<u>Note:</u> <u>Mero. fcn</u> does not have a well-defined residue.

Suppose a_{-1} is the residue of a mero. fcn. f at a point p. w.r.t. local coord. z,

Then, under change of coord $\tilde z = c_1 z + c_2 z^2 + \cdots$, $\tilde a_{-1} = c_1^{-1} a_{-1}$.

So, a_{-1} is <u>depends</u> on the coord.