Def: $p: \tilde{X} \longrightarrow X$ is called a <u>universal cover</u> if $\tilde{X}$ is conn. & simply conn. $(\pi_1 = 1)$.
conn.

Thm Every path-conn., locally path-conn., SLSC space admits a universal cover.

pf: Fix a pt $x_0 \in X$. Consider

- $\tilde{X} := \{$ homotopy classes of paths starting at $x_0 \}$.
- It admits a map
$$p: \tilde{X} \longrightarrow X$$
$$[\gamma] \longmapsto \gamma(1). \text{ endpoint of the path}$$

- Define <u>Topology on $\tilde{X}$</u>:

We'll define a topology on $\tilde{X}$ by declaring a <u>basis</u> $\mathcal{B}$ for the topology τ.

* Let τ be a topology on a set X. A subset $B \subseteq \tau$ is called a <u>basis</u> if every element of τ (i.e. an open set) can be represented as the union of some elements of $\mathcal{B}$.

Equivalent, $\forall\, U \underset{\text{open}}{\subseteq} X$ and $x \in U$, $\exists\, B \in \mathcal{B}$ s.t. $x \in B \subseteq U$.

<u>e.g.</u> $(\mathbb{R}$, Euclidean topology$)$, The set of <u>open intervals</u> is a basis.

* A basis B of a topology satisfies the following:

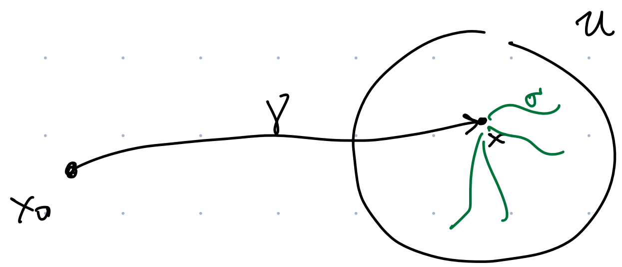
1) $\mathcal{B}$ covers X.
2) $\forall\, x \in B_1 \cap B_2$, $\exists\, x \in B_3 \subseteq (B_1 \cap B_2)$.

* Conversely, given a collection of subsets $\mathcal{B}$ of X, satisfying 1) and 2). One can define a topology τ of X consists of all subsets that are unions of elements in $\mathcal{B}$. This is a topology τ with basis $\mathcal{B}$.

* Therefore, if we find a collection of subsets satisfying 1) and 2), we can then define a topology.

- $\forall\ \gamma$ path starting from x_0, say end at x.

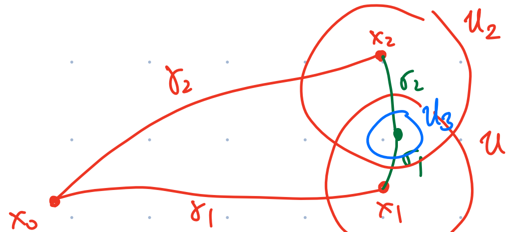
  Choose $\underset{\underset{x}{\Psi}}{U} \underset{\text{open}}{\overset{i}{\hookrightarrow}} X$ s.t. $i_* \pi_1(U) = \{e\} \subseteq \pi_1(X)$. (SLSC assumption)



Let $U_{[\gamma]} := \{ [\gamma \cdot \sigma] \mid \sigma: \text{path in } U, \text{ starting at } x \}$

Claim: $\{ U_{[\gamma]} \mid \gamma \text{ path from } x_0 \text{ to } x,\ U \ni x \text{ with SLSC} \}$ is a basis of $\tilde{X}$, i.e. satisfying 1) and 2) above.

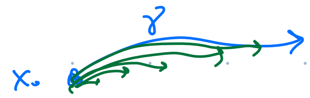
- 1) is clear.
- 2). Suppose $[\gamma] \in U_{1,[\gamma_1]} \cap U_{2,[\gamma_2]}$



- U_3 satisfies the SLSC property.
- $[\gamma] \in U_{3,[\gamma]}$: clear.
- $U_{3,[\gamma]} \subseteq U_{1,[\gamma_1]} \cap U_{2,[\gamma_2]}$.

  $([\gamma] \sim [\gamma_1 \cdot \sigma_1] \sim [\gamma_2 \cdot \sigma_2])$.

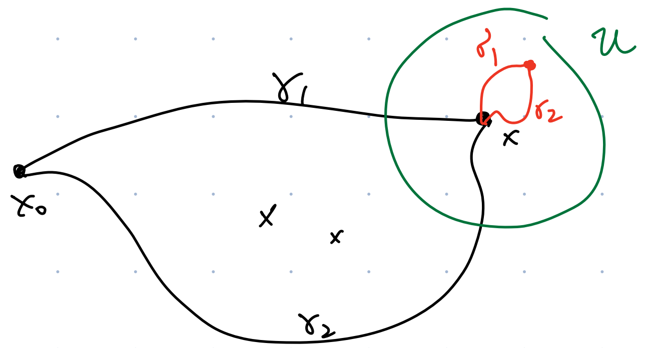
- $\tilde X$ is connected:



Let $[\gamma] \in \tilde X$, Consider $\gamma_s(t) := \gamma(st)$. $0 \le s \le 1$.

This connects $[\gamma]$ with the const. path at x_0.

- $\tilde X$ is Hausdorff: Suffices to show that for 2 paths γ_1, γ_2 w/ $\gamma_1(1) = \gamma_2(1) = x$, $[\gamma_1] \neq [\gamma_2]$. They have disjoint nbhd



Consider $\mathcal{U}_{[\gamma_1]}$ and $\mathcal{U}_{[\gamma_2]}$.

Suppose $\mathcal{U}_{[\gamma_1]} \cap \mathcal{U}_{[\gamma_2]} \neq \emptyset$. say $[\gamma_1 \cdot \sigma_1] = [\gamma_2 \cdot \sigma_2]$

$$\Rightarrow [\gamma_1] = [\gamma_2][\underbrace{\sigma_2 \cdot \sigma_1^{-1}}_{\text{in } i_*\pi_1(\mathcal{U}) \Rightarrow \text{ trivial}}] = [\gamma_2]. \quad \times. \quad \square$$

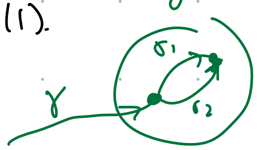
(Applying the SLSC condition.)

- $p: \tilde X \longrightarrow X$ is local homeom. ($\Rightarrow$ conti.)

$$\left(p|_{\mathcal{U}_{[\gamma]}} : \mathcal{U}_{[\gamma]} \xrightarrow{\sim} \mathcal{U} \right)$$

$$[\gamma \cdot \sigma] \longmapsto \sigma(1).$$

again uses SLSC property:



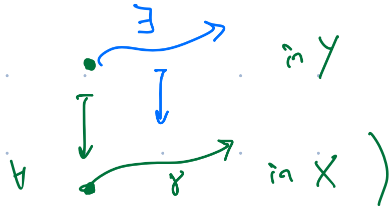
$[\gamma \cdot \sigma_1] = [\gamma \cdot \sigma_2]$

since $[\sigma_2 \cdot \sigma_1^{-1}]$ trivial

Fact: $p: Y \longrightarrow X$ local homeom.

Y: Hausdorff. X: connected, ..., SLSC, ... (same assumptions)

Suppose p has the curve lifting property, (i.e. $\forall$

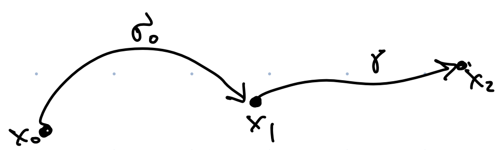


)

Then p is a covering map.

- Therefore to show that p is a covering map, it suffices to check the curve lifting property for $p: \tilde{X} \longrightarrow X$.

  * Let $\gamma: [0,1] \longrightarrow X$ be a curve with $\gamma(0) = x_1$.



  * $[\sigma_0] \in \tilde{X}$, $p([\sigma_0]) = x_1 = \gamma(0)$.

  * It's clear how to lift γ in $\tilde{X}$:

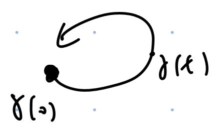
    - Let $\gamma_s(t) = \gamma(st)$, and consider $[\sigma_0 \cdot \gamma_s]_{0 \le s \le 1}$.

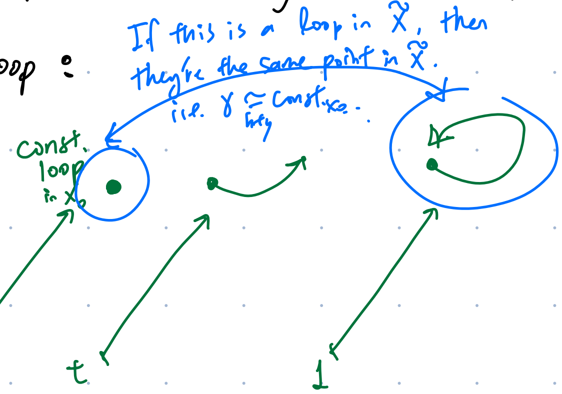
- $\underline{\pi_1(\tilde{X}) = \{e\}}$: Suffice to show: $p_* \pi_1(\tilde{X}, [\text{const}_{x_0}]) = 1$.

  i.e. loops in X that lift to loops in $\tilde{X}$ are htpy to const. loop.

- A loop in X lifts to the loop:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \arrow[ur] \arrow[r, "\gamma"] & X \end{tikzcd}$$





Coro: Every R.S. has a universal cover which is also a R.S.

(- mflds satisfy all the conditions like SLSC, ...)

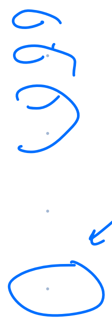
(- we showed last time that covering of a R.S. is also naturally a R.S.)

Prop: Suppose X satisfies the same top. conditions (e.g R.S.).

then $\forall H \subseteq \pi_1(X, x_0)$ subgp. $\exists$ covering space $p_H : X_H \longrightarrow X$

s.t. $p_{H*}(\pi_1(X_H, \tilde{x}_0)) = H$.

e.g. $\mathbb{R} \xrightarrow{\text{universal cover}} S^1$



Think of as $\mathbb{R}/\mathbb{Z}$.

$\pi_1(S^1) \cong \mathbb{Z}$.

To get $2\mathbb{Z} \subseteq \mathbb{Z}$;

we can identify $\alpha \sim \alpha + 2\mathbb{Z}$.

Idea of proof of Prop: Define an equivalence relation on $\tilde{X}$:

Say $[\gamma] \sim [\gamma']$ in $\tilde{X}$ if:
- $\gamma(1) = \gamma'(1)$ same endpoint
- $[\gamma \cdot \gamma'^{-1}] \in H$.

Define $X_H := \tilde{X}/\sim$

Check:
- $\tilde{X} \longrightarrow X$ descends to a map $X_H \longrightarrow X$.
- $X_H \longrightarrow X$ is a covering with the desired properties. $\square$

Def: Let $p: Y \to X$ be a covering. A <u>deck transformation</u> is a homeom. $f: Y \xrightarrow{\sim} Y$ s.t.

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

. (i.e. fiber-preserving)

They form a group, denoted $\mathrm{Deck}(Y \xrightarrow{p} X)$.

Def: A covering $p: Y \to X$ is <u>normal</u> (or <u>regular</u> or <u>Galois</u>) if $\forall\, y_1, y_2 \in Y$ with $p(y_1) = p(y_2)$, $\exists\, f \in \mathrm{Deck}(Y \xrightarrow{p} X)$ s.t. $f(y_1) = f(y_2)$.

Fact: The <u>universal cover</u> is normal, with $\mathrm{Deck}(\tilde{X} \to X) \cong \pi_1(X)$.

e.g. $\exp: \mathbb{C} \to \mathbb{C}^\times$ is the universal covering of $\mathbb{C}^\times$.

There are deck transformations:

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "z \mapsto z+2\pi i n"] \arrow[dr, "\exp"'] & & \mathbb{C} \arrow[dl, "\exp"] \\ & \mathbb{C}^\times & \end{tikzcd} \qquad \forall n \in \mathbb{Z}.$$

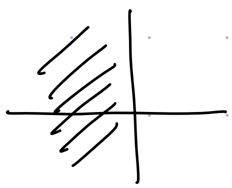
(In fact, these are all the deck transf. of exp.)

$$\mathrm{Deck}(\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times) = \{\tau_n : z \mapsto z + 2\pi i n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}. \quad \cong \pi_1(\mathbb{C}^\times)$$

e.g. $\mathbb{D}^\times = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$

$\uparrow \exp$

$\mathbb{H}$



$$\mathrm{Deck}(\mathbb{H} \xrightarrow{\exp} \mathbb{D}^\times) \cong \pi_1(\mathbb{D}^\times)$$

$$= \{ z \mapsto z + 2\pi i n \mid n \in \mathbb{Z} \}$$

$$\cong \mathbb{Z}$$

e.g. $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ lattice in $\mathbb{C}$, $\quad \mathbb{C} \to \mathbb{C}/\Lambda$ is the universal cover.

$$\mathrm{Deck}(\mathbb{C} \to \mathbb{C}/\Lambda) = \{\text{translations by } \lambda \in \Lambda\} \cong \Lambda \cong \mathbb{Z} \times \mathbb{Z}.$$

$$\cong \pi_1(\mathbb{C}/\Lambda)$$

§ Finite gp actions on R.S.

- G. finite gp, acts holomorphically on a R.S. X:

$$G \times X \longrightarrow X \quad \text{gp action, where } g: x \mapsto g\cdot x \in \underset{\text{holo. auto.}}{\mathrm{Aut}(X)}$$

- Kernel of the action is $\ker := \{ g \in G \mid g\cdot x = x \ \forall x \in X \} \subseteq G$.

<u>Note</u>: $\ker \subseteq G$ is a normal subgp., and $G/\ker \curvearrowright X$ has exactly the same orbits, with trivial kernel.

So, we usually assume that the ker is trivial.

- The <u>quotient space</u> X/G is the set of <u>orbits</u>.

$$\pi: X \longrightarrow X/G$$
$$x \longmapsto \mathrm{orb}(x)$$

The natural <u>quotient topology</u>: $U \subseteq X/G$ is called <u>open</u> if $\pi^{-1}U$ is open.

§ Equip X/G with a R.S. str.,

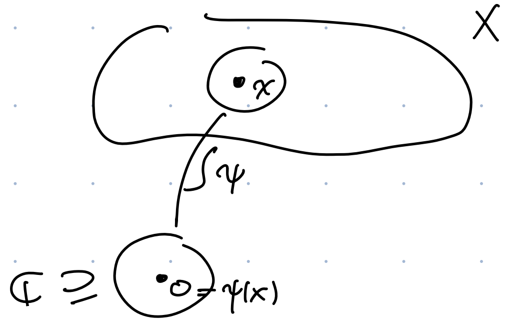
<u>Note</u>: We've discussed earlier the case when each $g \in G \setminus \{e\}$ has no fixed point.

i.e. each $x \in X$ has no nontrivial stabilizer. It turns out that, we still can equip X/G w/ a R.S. str., even if the action has fixed points.

<u>Lemma</u>: Let $x \in X$ and $G_x \subseteq G$ be the stabilizer subgp.

Then G_x must be a finite <u>cyclic</u> gp.

pf: Let $g \in G_x \setminus \{e\}$. Choose a local coordinate at x:



In the local coord., g has a power series expansion:

$$g(z) = \underbrace{a_1(g)}_{\neq 0} z + a_2(g) z^2 + \cdots \quad (g(0) = 0, \text{ so the const. term} = 0).$$

Since g is an automorphism.

Define the fcn: $G_x \xrightarrow{a_1} \mathbb{C}^\times$

$g \longmapsto a_1(g)$.

This is a gp homom:

$$g(h(z)) = g\left(a_1(h) z + a_2(h) z^2 + \cdots\right)$$

$$= a_1(g)\left(a_1(h) z + a_2(h) z^2 + \cdots\right) + a_2(g)\left(a_1(h) z + \cdots\right)^2 + \cdots$$

$$= a_1(g)\, a_1(h)\, z + O(z^2).$$

$$\Longrightarrow \quad a_1(gh) = a_1(g)\, a_1(h).$$

Note: Every finite subgp. of $\mathbb{C}^\times$ is cyclic.

So, it suffices to show: $G_p \xrightarrow{a_1} \mathbb{C}^\times$ is injective

Suppose $g \in G_p$ s.t. $a_1(g) = 1 \in \mathbb{C}^\times$.

Then: $$g(z) = z + a_2(g) z^2 + \cdots$$

Assume that $g \neq e$. then $a_2(g), a_3(g), \ldots$ not all 0.

Choose smallest $m \geq 2$ with $a_m(g) \neq 0$.

Then $$g(z) = z + \underset{\neq 0}{a_m(g)} z^m + \cdots$$

$$\Rightarrow \quad g(g(z)) = (z + a_m(g) z^m) + a_m(g) (z + a_m(g) z^m)^m + \cdots$$
$$= z + 2 a_m(g) z^m + \cdots$$
$$g^k(z) = z + k a_m(g) z^m + \cdots$$

Contradicts with $g \in G_p$ is of finite order. $\square$

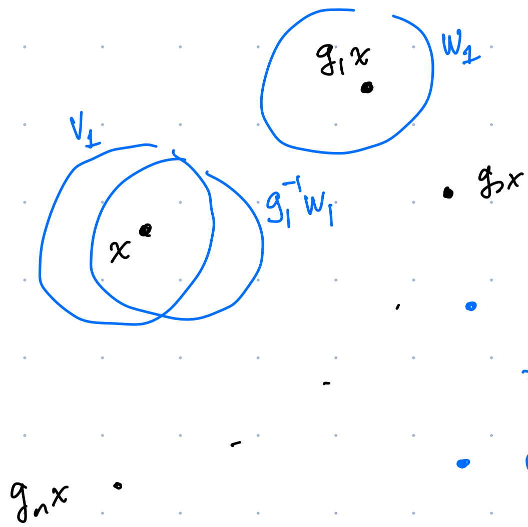
Note: $\overset{\text{finite}}{G} \curvearrowright X$. The points of X with nontrivial stabilizers are discrete.

(follows from the identity thm.)

Prop: $\forall x \in X$, $\exists\, \overset{x}{\underset{\text{open}}{\overset{\cap}{U}}} \subseteq X$ s.t.

1) U is invar. under the stabilizer G_x. (i.e. $g \cdot u \in U \quad \forall g \in G_x, u \in U$).
2) $U \cap g \cdot U = \phi \quad \forall g \notin G_x$.
3) $U/G_x \longrightarrow X/G$ is a homeom. onto an open subset of X/G.
4) No point in U except x is fixed by any elt of G_x.

pf: Let $G / G_x = \{g_1, \ldots, g_n\}$ Hausdorff $\Rightarrow$ find separating open sets



- $R_i := V_i \cap (g_i^{-1} W_i)$

  $\Longrightarrow$ R_i and $g_i R_i$ disjoint.

- $R := \bigcap R_i$.

- $U := \bigcap_{g \in G_p} g \cdot R$

$\Longrightarrow$ 1), 2) are clear.

- $U / G_x \longrightarrow X / G$. clearly injective.

This leads to a construction of R.S. str. on X/G. Locally, an open set is isom. to U / G_x.

1) If $G_x = \{e\}$, then locally it's just $U \subseteq X$, which can be given a chart from X.

2) If $G_x \neq \{e\}$, then $G_x \cong \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 2$.

<u>Fact:</u> $\exists$ a local coordinate at x so that $g \in G_x$ is of the form:

$g(z) = \exp\left(\frac{2\pi i}{n} k\right) z$ (a "local normal form")

$\mathbb{D}_0(r) / \mathbb{Z}_n \xrightarrow[\sim]{z \mapsto z^n} \mathbb{D}_0(r^n)$ then gives a chart near x.

$$\begin{array}{c} X \quad \text{cpt. R.S.} \\ \Big\downarrow \pi \\ X/G \end{array}$$

- Crit. points = points in X w/ nontrivial stabiliser.
- Crit. values = Orbits w/ nontrivial stabiliser
- degree $(\pi) = |G|$.

- $\forall$ critical value $[x] \in X/G$, $\exists\, r \geq 2$ (multiplicity) s.t. each preimage of $[x]$ has mult. r $(= |G_x|)$ and $\pi^{-1}([x])$ consists of $|G|/r$ points (# pts in the orbit.)

- By Riemann–Hurwitz, we have:

k many crit. values

$$2 - 2g(X) = |G|\,(2 - 2g(X/G)) - \sum_{i=1}^{k} \frac{|G|}{r_i}(r_i - 1)$$

another cpt. R.S.

$$= |G| \cdot \left[-2g(X/G) + 2 - \sum_{i=1}^{k} \left(1 - \frac{1}{r_i}\right) \right]$$

<u>Elementary lemma:</u> Let $R = \sum_{i=1}^{k} \left(1 - \frac{1}{r_i}\right)$ for some $r_i \geq 2$.

- $R < 2 \iff \begin{cases} k = 1, 2, \text{ or} \\ k = 3 \text{ and } r_i\text{'s are: } \{2,2,r_3\}, \\ \{2,3,3\},\ \{2,3,4\},\ \{2,3,5\}. \end{cases}$

- $R = 2 \iff \begin{cases} k = 3,\ r_i\text{'s are: } \{2,3,6\}, \{2,4,4\}, \{3,3,3\} \\ k = 4,\ r_i\text{'s are: } \{2,2,2,2\}. \end{cases}$

- $R > 2 \implies R \geq 2 + \frac{1}{42}.$

Applications:

- Finite gp of holo. auto. on $\mathbb{CP}^1$: $g(\mathbb{CP}^1)=0$, LHS $=2$.

  $\Rightarrow g(\mathbb{CP}^1/G)=0$,

  and $2 = |G|\cdot\left[2-\sum_{i=1}^{k}\left(1-\frac{1}{r_i}\right)\right] \Rightarrow R<2$.

$\rightarrow$ only possibilities: $k=2 \rightarrow G$ cyclic

$k=3$, $r=\{2,2,r_3\} \longrightarrow G \cong D_{2r}$.

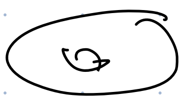
$r=\{2,3,3\} \longrightarrow G \cong A_4$

$r=\{2,3,4\} \longrightarrow G \cong S_4$

$r=\{2,3,5\} \longrightarrow G \cong A_5$

$\left.\begin{matrix} A_4 \\ S_4 \\ A_5 \end{matrix}\right\}$ Sym. gp of platonic solids.

This is actually the full list of finite subgp. of $SO(3,\mathbb{R})$.

- Finite gp action on : $g=1$.

$$\Rightarrow -2g(X/G)+2-\sum_{i=1}^{k}\left(1-\frac{1}{r_i}\right)=0$$

  - If $g(X/G)=1$, then $R=0 \Rightarrow$ no branching.

    $\Rightarrow$ any g has no fixed pt. on $\mathbb{C}/\Lambda \Longrightarrow$ translations.

  - If $g(X/G)=0$, then $R=2$, and there could be 4 possible ramification profiles.

<u>Hurwitz thm</u>: G finite $\overset{\text{holo.}}{\curvearrowright} X$, $g(X) \geq 2$.

Then $|G| \leq 84(g-1)$.

(In fact, one can show that $|\mathrm{Aut}(X)| < +\infty$.)

<u>pf</u>: Recall $\underset{0>}{2-2g} = |G| \left(-2g(X/G) + 2 - R \right)$

- If $g(X/G) \geq 1$,
  - if $R=0$, then $g(X/G) \geq 2$, and $|G| \leq g-1$.
  - if $R \neq 0$, then $R \geq \frac{1}{2}$, $2g(X/G) - 2 + R \geq \frac{1}{2}$.

    $\Rightarrow |G| \leq 4(g-1)$.

- If $g(X/G) = 0$, then $2g-2 = |G| \cdot (R-2)$.

  $\Rightarrow R > 2 \Longrightarrow R \geq 2 + \frac{1}{42}$. $\Rightarrow |G| \leq 84(g-1)$.

<u>Note</u>: The bound can be achieved in some genus.

$g=3$. $\underset{\substack{\uparrow \\ \text{Klein quartic}}}{\mathrm{Aut}(X_3)} \cong PSL_2(\mathbb{F}_7)$ has order 168.

$g=7$, $\mathrm{Aut}(X_7) \cong PSL_2(\mathbb{F}_8)$ has order 504