Rmk: More sources of Riemann surfaces:

- **Covering Spaces:** $p: Y \longrightarrow X$ is a covering space if $\forall x \in X$, $\exists_x \in \mathcal{U} \subseteq X$ s.t. $p^{-1}\mathcal{U} = \coprod_{\alpha \in I} V_\alpha$, where $V_\alpha \xrightarrow[\cong \text{ homeom.}]{p|_{V_\alpha}} \mathcal{U}$

  - If is a R.S., then can be equipped w/ a R.S. str.

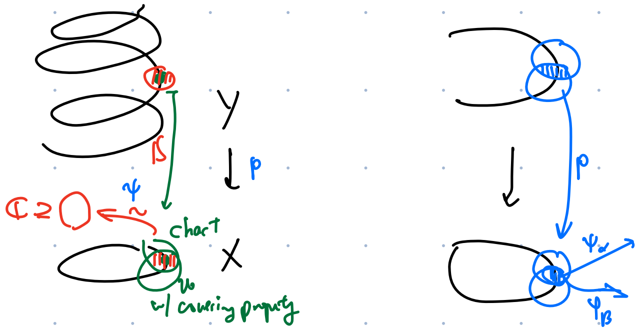


chart $\psi \circ p$

holomorphicity of:
$(\psi_\alpha \circ p) \circ (\psi_\beta \circ p)^{-1} = \psi_\alpha \circ \psi_\beta^{-1}$. $\square$

  - There is a notion of universal cover, which governs all covering space.
    When X is connected (and locally simply connected).,

$$\{\text{subgps of } \pi_1(X)\} \xleftrightarrow{1-1} \{\text{path-connected covering } Y \longrightarrow X\}.$$

$$\begin{array}{ccc}
\{1\} & \longleftrightarrow & \text{"}\tilde{X}\text{" universal cover of } X \\
| & & | \\
H & & \text{"}\tilde{X}/H\text{"} \\
| & & | \\
\pi_1(X) & \longleftrightarrow & X
\end{array}$$

$$\{\text{normal subgp of } \pi_1(X)\} \longleftrightarrow \{\text{normal/Galois covering } Y \to X\}$$

(A deck transformation $p. \; Y \longrightarrow X$ is a homeom. $Y \xrightarrow{\sim} Y$ with p to X from both, commuting ($\circlearrowleft$).

If $\mathrm{Deck}(p)$ acts transitively on the fibers, we say the cover is regular or normal or Galois.)

Analogy: Let K/F be a finite Galois extension of fields.

$$\{\text{subgps of } Gal(K/F)\} \longleftrightarrow \{\text{intermediate fields } K/E/F\}$$
$(K \to K \text{ id on } F)$

$$\begin{array}{ccc} \{1\} & \longleftrightarrow & K \\ | & & | \\ H & \longleftrightarrow & K^H \\ | & & | \\ Gal(K/F) & \longleftrightarrow & F \end{array}$$

$$\{\text{normal subgp of } Gal(K/F)\} \longleftrightarrow \{\text{normal / Galois ext. } E/F\}.$$

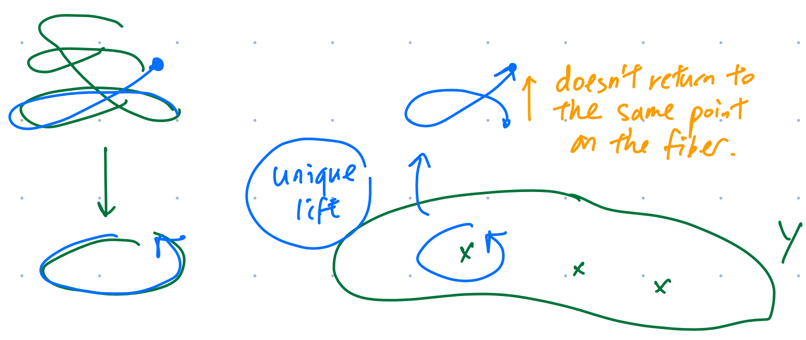
* When X is a R.S., covering spaces are also R.S., and deck transformations are biholomorphic. So the whole story of topological spaces & covering maps specialized nicely to R.S. & holo. maps.

Recall from last time: $f: X \longrightarrow Y$ non const holo. map btw cpt. conn. R.S.

$\rightsquigarrow$
- a discrete set of critical values $\Delta := f(Crit(f)) \subseteq Y$.
- monodromy at any $y_0 \in Y \setminus \Delta$:

$$\pi_1(Y \setminus \Delta, y_0) \longrightarrow Aut(\pi_1^{-1}(y_0)) = \{\text{Bijection } \pi_1^{-1}(y_0) \to \pi_1^{-1}(y_0)\}$$

Locally, at a branched point:



Riemann existence thm: Y: conn. R.S. $\Delta \subseteq Y$ discrete. $y_0 \in Y \setminus \Delta$

$\rho: \pi_1(Y \setminus \Delta) \longrightarrow S_d$ transitive

Then $\exists$ conn. R.S. X, degree d holo. $f: X \longrightarrow Y$ w/ Crit value Δ and Monodromy ρ.

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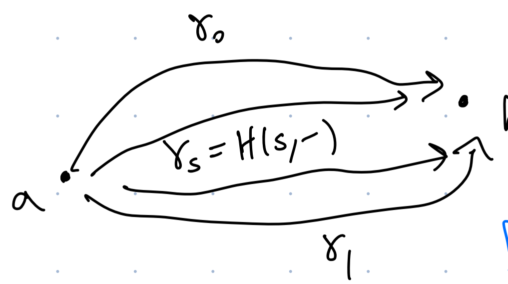
§ Fundamental group.

Def: X: top. space, $a, b \in X$. Two paths $\gamma_0, \gamma_1 : [0,1] \longrightarrow X$ from a to b: (i.e. $\gamma_0(0) = \gamma_1(0) = a$, $\gamma_0(1) = \gamma_1(1) = b$) are **homotopic** if $\exists$ $H: [0,1] \times [0,1] \longrightarrow X$ conti.

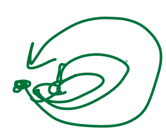
s.t.
- $H(0,t) = \gamma_0(t) \quad \forall t \in [0,1]$
- $H(1,t) = \gamma_1(t) \quad \forall t \in [0,1]$
- $H(s,0) \equiv a$, $H(s,1) \equiv b$.

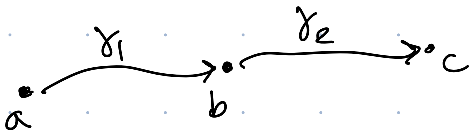


i.e. $\{\gamma_s\}_{s \in [0,1]}$ is a conti. deformation of γ_0 to γ_1.

Denote: $\gamma_0 \underset{htpy}{\simeq} \gamma_1$.

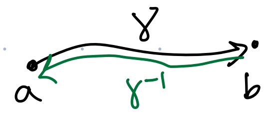
Rmk:
- Homotopy defines an equivalence relation on the set of all paths from a to b.
- A closed curve (i.e. loop) $(\gamma(0) = \gamma(1) = a)$ is null-homotopic if it's homotopic to the const. curve at a.



Def: 

Product curve: $\gamma_1 \cdot \gamma_2$ defined as:

$$(\gamma_1 \cdot \gamma_2)(t) := \begin{cases} \gamma_1(2t), & 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}.$$

Def: 

Inverse γ^{-1}: $\gamma^{-1}(t) := \gamma(1-t)$.

Ex: Both are well-defined on homotopy classes.

Def/Thm: X - top. space, $a \in X$.

$\pi_1(X, a) := \{$homotopy classes of closed curves in X based at $a\}$

forms a group. (with the product and inverse as above).

Called: Fundamental group of X based at a.

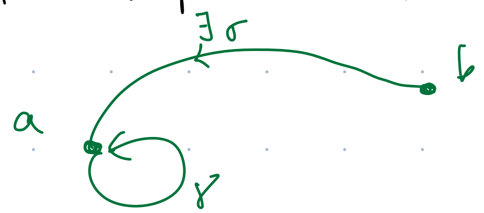
Rmk: Associativity does not hold directly on products:

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3) \neq (\gamma_1 \cdot \gamma_2) \cdot \gamma_3,$$

but they're homotopic: $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3) \underset{\text{htpy}}{\simeq} (\gamma_1 \cdot \gamma_2) \cdot \gamma_3$.

Similarly, $\gamma \cdot \gamma^{-1} \neq \text{const}_a$, but $\gamma \cdot \gamma^{-1} \underset{\text{htpy}}{\simeq} \text{const}_a$.

Rmk: If X is path-connected, then $\pi_1(X, a) \cong \pi_1(X, b)$:



$\gamma \mapsto \sigma \cdot \gamma \cdot \sigma^{-1}$

Def: A path-connected space X is called <u>simply connected</u> if $\pi_1(X)$ is trivial.

<u>e.g.</u>
- convex subsets in $\mathbb{R}^n$ are simply connected
- $\mathbb{C}P^1$ is simply connected
- $\mathbb{C}/\Lambda$ is <u>not</u> simply connected.

<u>Functoriality</u>: $f: X \longrightarrow Y \quad \rightsquigarrow \quad f_*: \pi_1(X, a) \longrightarrow \pi_1(Y, f(a))$
$\qquad a \longmapsto f(a)$

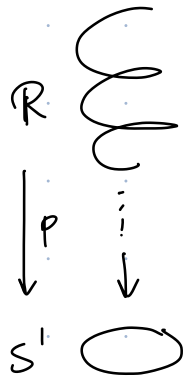
$$X \xrightarrow{f} Y \xrightarrow{g} Z \quad \rightsquigarrow \quad (g \circ f)_* = g_* \circ f_*.$$

* There is a <u>functor</u>:

$$\left\{ \begin{array}{c} \text{pointed topological} \\ \text{spaces } (X, x_0) \end{array} \right\} \xrightarrow{\pi_1} \{ \text{Groups} \}$$

§ $\pi_1(S^1) \cong \mathbb{Z}$.

We'll "lift" loops on S^1 to paths in $\mathbb{R}$:



$\hookrightarrow \mathbb{R}^3$ by $s \longmapsto (\cos(2\pi s), \sin(2\pi s), s)$.

$$\Big\downarrow p$$

$(\cos(2\pi s), \sin(2\pi s), 0)$.

Def: Let $\tilde{X} \xrightarrow{p} X$ be a covering map (e.g. $p: \mathbb{R} \to S^1$).

$$\begin{array}{c} X \\ \uparrow f \text{ (conti.)} \\ A \end{array}$$

A (conti.) map $\tilde{f}: A \to \tilde{X}$ is a lift of f if

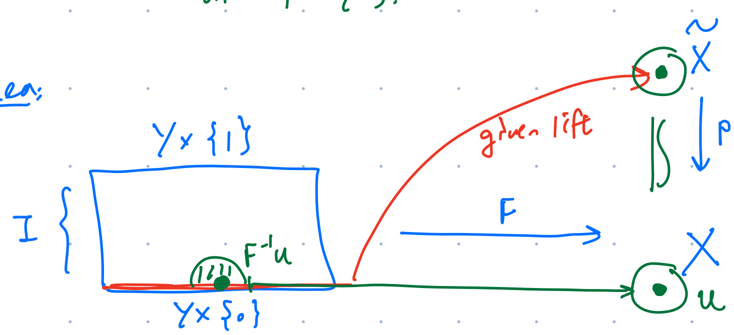
$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ A \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}.$$

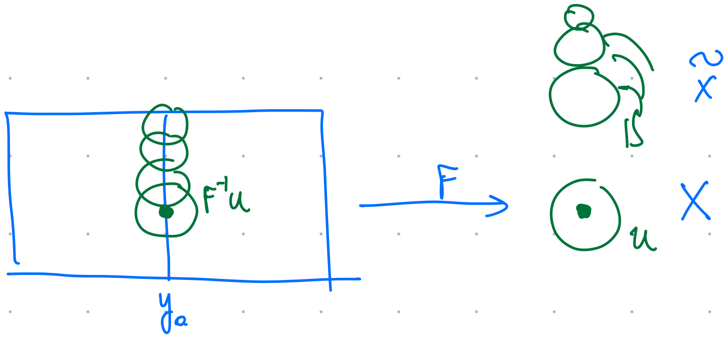
Homotopy Lifting Property: $p: \tilde{X} \to X$ covering.

Let $F: Y \times I \to X$ be a (conti.) map, and let $\tilde{F}: Y \times \{0\} \to \tilde{X}$ be a lift of $F|_{Y \times \{0\}}$.

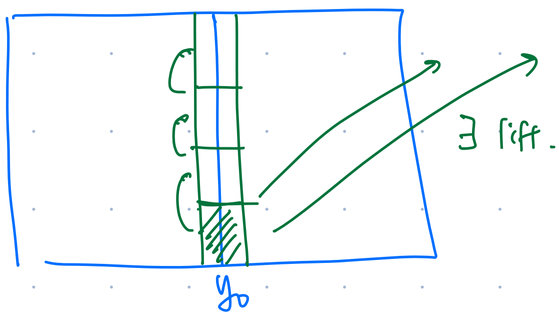
Then $\exists!$ $\tilde{F}: Y \times I \to \tilde{X}$ lifting of F restricting to the given lift on $Y \times \{0\}$.

Idea:





Since $\{y_0\} \times I$ is cpt, $\exists$ a finite cover of such open sets $F^{-1}u$.



Claim: The lifting is unique in the case that $Y = \{pt\}$ (e.g. y_0).
(given 2 coverings, find common evenly covering nbhd, $\leadsto$ uniqueness)

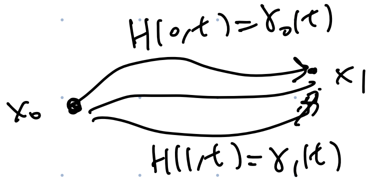
$\Longrightarrow$ such $\tilde{F}$ exists and is unique. $\square$

Coro: $\forall$ path $f : I \longrightarrow X$ w/ $f(0) = x_0 \in X$, $\forall \tilde{x}_0 \in p^{-1}(x_0)$,
$\exists!$ lift $\tilde{f} : I \longrightarrow \tilde{X}$ s.t. $\tilde{f}(0) = \tilde{x}_0$.

$$\begin{array}{ccc} & & \tilde{X} \\ & & \downarrow p \\ I & \xrightarrow{f} & X \end{array}$$

Unique lift of path, after specifying the starting point $\tilde{x}_0$

<u>Coro:</u> Let $H: I \times I \longrightarrow X$ be a homotopy of paths.
(s,t)

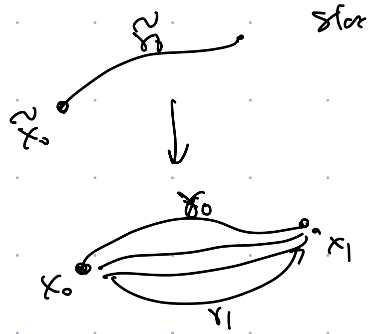


$H(s,0)=x_0$
$H(s,1)=x_1$

$$\begin{array}{ccc} & \tilde{X} & \forall \tilde{x}_0 \\ & \downarrow p & \downarrow \\ I \times I & \xrightarrow{H} X & x_0 \end{array}$$

$\exists!$ lifted homotopy $\tilde{H}: I \times I \longrightarrow \tilde{X}$ of paths w/ $\tilde{H}(s,0) = \tilde{x}_0$.

<u>pf:</u> By previous Coro., $\exists!$ lift of $H(0,t) = \gamma_0(t)$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$.



By Homotopy Lifting Property, $\exists!$ lift $\tilde{H}: I \times I \longrightarrow \tilde{X}$.

<u>Need to make sure:</u> the endpoints of $\tilde{\gamma}_s$ are the same!

- $H(s,0) = x_0$ is a const. path in s. lift uniquely to a path in $\tilde{X}$, therefore must be the const. path at $\tilde{x}_0$
- Similarly, $H(s,1) = x_1$ is a const. path, therefore, the (unique) lift must be a const. path. thus $\tilde{H}(s,1)$ = the endpoints of $\tilde{\gamma}_s$ are the same. $\square$

Thm: $\pi_1(S^1) \cong \mathbb{Z}$.

pf: Let $f: I \to S^1$ be a loop based at $(1,0) \in S^1$.

(its homotopy class is an elt. of $\pi_1(S^1)$.)

By Coro., $\exists!$ lift of f: $\tilde{f}: I \longrightarrow \mathbb{R} \cong \{(\cos(2\pi s), \sin(2\pi s), s)\}$

$0 \longmapsto 0 \leftrightarrow (1,0,0)$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Note that $p^{-1}((1,0)) = \mathbb{Z} \subseteq \mathbb{R}$.

Therefore, the endpoint of the lift $\tilde{f}(1)$ must be an integer $n \in \mathbb{Z}$.

Claim: $\pi_1(S^1) \longrightarrow \mathbb{Z}$ is an isomorphism.

$[f] \longmapsto \tilde{f}(1)$

* well-defined: If $f \underset{\text{htpy}}{\sim} f'$, by Prop., $\tilde{f} \underset{\substack{\text{htpy} \\ \text{of paths}}}{\sim} \tilde{f}'$, thus have $\tilde{f}(1) = \tilde{f}'(1)$.

* (Ex): gp homom, injective, surjective.

e.g. $S^1 \xrightarrow{P_2} S^1$ (2-sheeted cover).

$z \longmapsto z^2$

$(P_2)_*: \pi_1(S^1) \longrightarrow \pi_1(S^1)$ is the map $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$.

Prop: Let $p: \tilde{X} \longrightarrow X$ be a covering

$\tilde{x}_0 \longmapsto x_0$

Then: $P_*: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is injective,

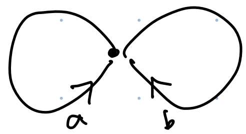
Moreover, the image consists of loops in (X, x_0) that lifts to loops in $\pi_1(\tilde{X}, \tilde{x}_0)$

pf. • Suppose $\tilde{f}_0: I \longrightarrow \tilde{X}$ is a loop s.t. $f_0 = p \circ \tilde{f}_0$ is htpy to const. path.

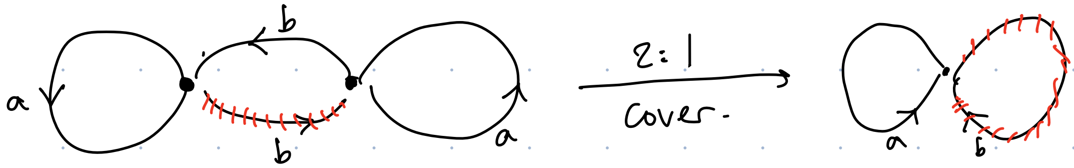
By Homotopy Lifting Property, $\tilde{f}_0$ is htpy to a const. path. thus trivial in π_1. $\square$

• The second statement is clear.

e.g: $S^1 \vee S^1$



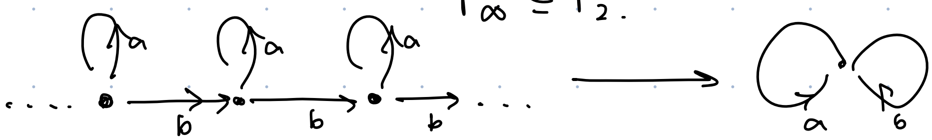
$\pi_1(S^1 \vee S^1)$ is a free group $F_2 \cong \langle a, b \mid - \rangle$.



$\Rightarrow$ F_3 is a subgp of F_2!

Similarly, one can construct subgp of F_2 isom. to F_n.

Even more: $F_\infty \subseteq F_2$.



Q: Is every subgp of $\pi_1(X, x_0)$ given by the image of $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ of some covering $p: \tilde{X} \to X$?

Start with the case of $\{1\} \subseteq \pi_1(X, x_0)$: (turn out to be the hardest case)

" $\exists$ covering $\tilde{X} \to X$ st. $\pi_1(\tilde{X}) \cong \{e\}$? "

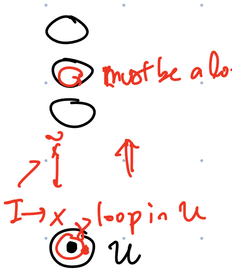
<u>Note:</u> Need some conditions on X:

Suppose $X, \tilde{X}$ connected, $p: \tilde{X} \to X$ covering, $\pi_1(\tilde{X}) = \{e\}$.

Then: $\tilde{X}$

$\downarrow$

X



$\Longrightarrow$ Contracts to a point in $\tilde{X}$ (not necessarily within the open set) since $\pi_1(\tilde{X}) = 1$.

$\Downarrow$

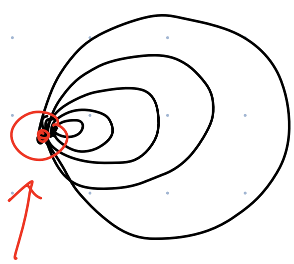
γ contracts to a pt. in X

<u>Def:</u> X is <u>semi-locally simply-connected</u> (SLSC) if.

$\forall x \in X$, $\exists \underset{\substack{\Psi \\ x}}{\mathcal{U}} \overset{i}{\underset{\text{open}}{\hookrightarrow}} X$ s.t. $i_*: \pi_1(\mathcal{U}, x) \longrightarrow \pi_1(X, x)$ is trivial.

<u>e.g.</u> All CW cpx are SLSC., R.S.s are SLSC.

<u>Non-e.g.:</u>



w/ top. induced from $\mathbb{R}^2$.

no nbhd $\mathcal{U} \overset{i}{\hookrightarrow} X$ s.t. $i_{\mathcal{U}*}$ is trivial.