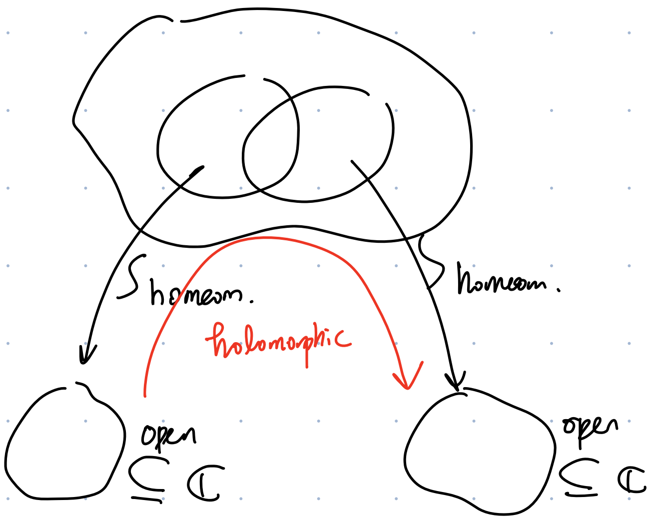
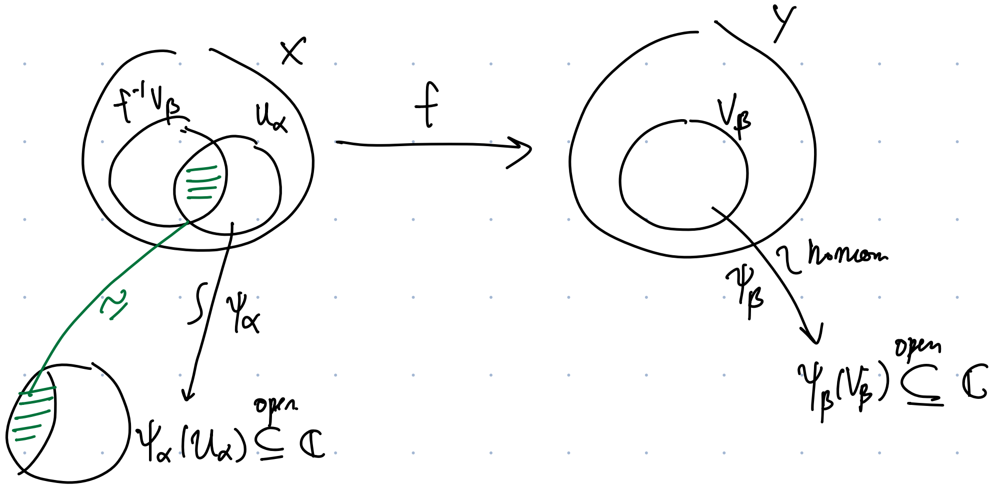
Recap: Last time: Define R.S., & examples ($\mathbb{CP}^1$, Alg curve, Torus, Quotients)



Recall $f: X \to Y$ is holomorphic if f is continuous and:



$$\begin{tikzcd} \varphi_\alpha(U_\alpha \cap f^{-1}V_\beta) \arrow[r, "\varphi_\alpha^{-1}", "\sim"'] \arrow[rrr, bend right] & U_\alpha \cap f^{-1}V_\beta \arrow[r, "f"] & V_\beta \arrow[r, "\psi_\beta", "\sim"'] & \psi_\beta(V_\beta) \end{tikzcd}$$

$\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is holomorphic $\forall\, \alpha, \beta$.

Rmk: Many classical results on holo. fcns hold for holo. maps between R.S.s.

e.g. (Identity thm). $\mathcal{U} \subseteq \mathbb{C}$ open, connected subset; $f, g: \mathcal{U} \xrightarrow{\text{holo.}} \mathbb{C}$

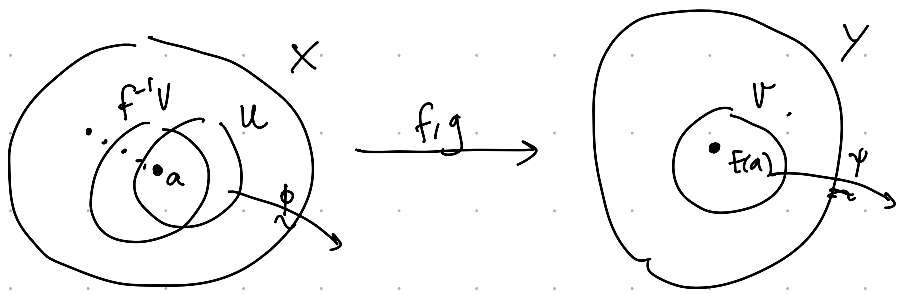
If $S := \{z \in \mathcal{U} \mid f(z) = g(z)\}$ contains a limit point, then $f \equiv g$.

(Rmk: Recall that this doesn't hold for smooth fcns in $\mathbb{R}$.)

Prop: X, Y: R.S., X: connected, $f, g: X \longrightarrow Y$ holo.

If $S := \{z \in \mathcal{U} \mid f(z) = g(z)\}$ has a limit point (in X), then $f \equiv g$.

pf:



Let $a \in X$ be a limit pt of S.

Apply the identity thm to the holo. fcn:

$$\phi(\mathcal{U} \cap f^{-1}V) \xrightarrow{\psi \circ (f, g) \circ \phi^{-1}} \psi(V).$$

$\Rightarrow f \equiv g$ on $\mathcal{U} \cap f^{-1}V$.

Consider $Z := \{x \in X \mid f \equiv g \text{ in an open nbhd of } x\} \subseteq X$.

- $Z \neq \phi$. (we've just proved that $a \in Z$).
- Z is open (clear).

- Z is closed: Suppose $b \in \overline{Z}$ is a limit pt of Z.
  Then the above argument shows that $f \equiv g$ in a nbhd of b. Thus $b \in Z$ by definition.

$\Longrightarrow$ $Z = X$ since X is connected. $\square$

Recall: $f : \underset{\text{open}}{U} \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ holo. Suppose $f(z_0) = 0$ is a zero.
Then $\exists\, k \geq 1$, $z_0 \in V \underset{\text{open}}{\subseteq} U$, $g : V \longrightarrow \mathbb{C}$ holo., $g(z_0) \neq 0$.
s.t. $f(z) = z^k g(z)$ on V.

Here, k is the order of the zero z_0.

Prop (Local normal form of holo. map). X, Y: R.S..
Let $f : X \longrightarrow Y$ be a nonconstant holo. map
$a \longmapsto b$.

Then $\exists\, k \geq 1$, $U \xrightarrow[\sim]{\phi} \phi(U) \subseteq \mathbb{C}$, $V \xrightarrow{\psi} \psi(V) \subseteq \mathbb{C}$
$a \longmapsto 0$, $b \longmapsto 0$

s.t. $f(U) \subseteq V$, and

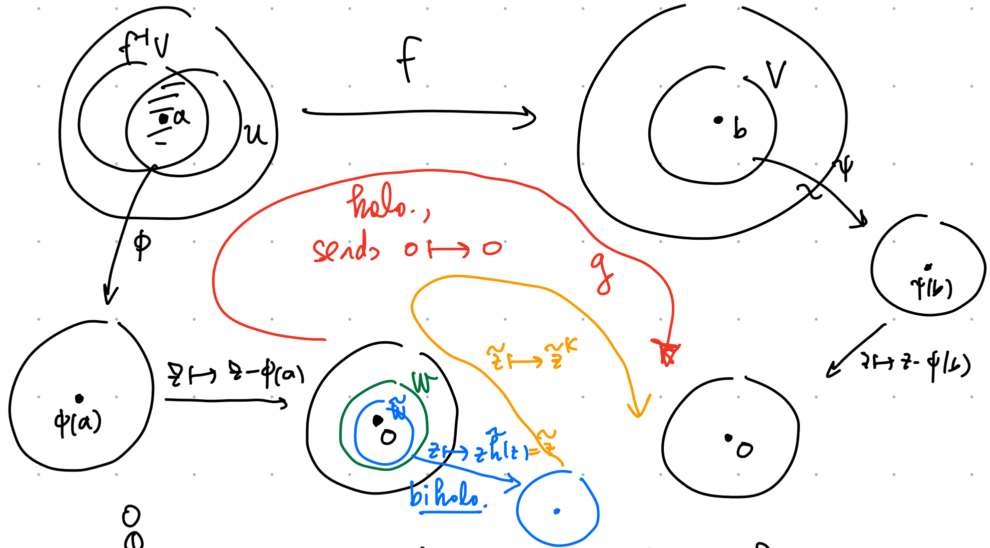
$$\begin{array}{ccccccc} \phi(U) & \xrightarrow{\phi^{-1}} & U & \xrightarrow{f} & V & \xrightarrow{\psi} & \psi(V) \\ 0 & \longmapsto & a & \longmapsto & b & \longmapsto & 0 \end{array}$$

is $z \longmapsto z^k$.

(Ex.)

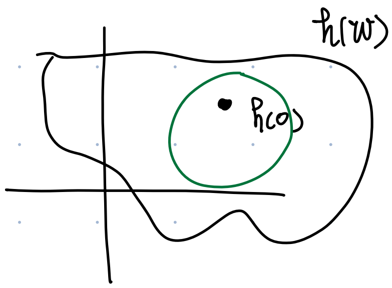
Rmk: k is called the multiplicity of f at a, and is indep. of choice of charts.

pf:



$\exists\, k \geq 1,\ 0 \in W \subseteq U \cap f^{-1}V,\ h: W \to \mathbb{C}$ holo, $h(0) \neq 0$

s.t. $g(z) = z^k h(z)$ on W.



$\exists\, \tilde{W} \underset{\text{open}}{\subseteq} W$ s.t. h admits k^{th} root in $\tilde{W}$.

i.e. $\exists\, \tilde{h}: \tilde{W} \to \mathbb{C}$ holo. $\tilde{h} \neq 0$ on $\tilde{W}$.

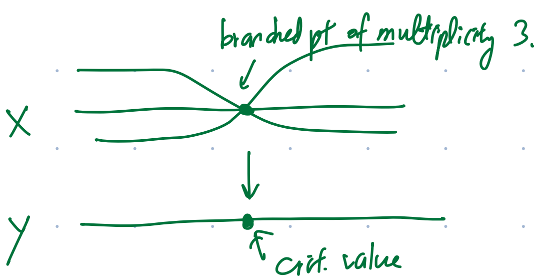
s.t. $h(z) = \tilde{h}(z)^k$ on $\tilde{W}$.

$\Rightarrow$ On $\tilde{W}$, we have $g(z) = (z \cdot \tilde{h}(z))^k$.

After the change of variable $\tilde{z} := z\tilde{h}(z)$,

the composition becomes $\tilde{z} \mapsto \tilde{z}^k$. $\square$

Rmk: Picture:



Def: $x \in X$ is called a critical point of f (or branched point) if the corresponding $k > 1$.

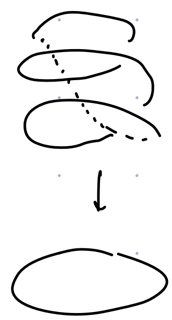
- The set of such points is denoted $\mathrm{Crit}(f) \subseteq X$ (Ex: It is discrete in X)
- A critical value is a point in $f(\mathrm{Crit}(f))$. Points in $Y \setminus f(\mathrm{Crit}(f))$ are called regular value.
- $f: X \longrightarrow Y$ is a "branched cover" if it has at least 1 branched points.

Rmk: Later, we'll discuss the notion of covering map:

$f: X \longrightarrow Y$ is a covering map if $\forall y \in Y$, $\exists\ \underset{\text{open}}{\mathcal{U} \subseteq Y}$ with $y \in \mathcal{U}$

s.t. $f^{-1}\mathcal{U} = \coprod_{\alpha \in I} V_\alpha$, each $V_\alpha \underset{\text{open}}{\subseteq} X$ and $V_\alpha \xrightarrow[f]{\overset{\text{homeom}}{\sim}} \mathcal{U}$. $\forall \alpha$.

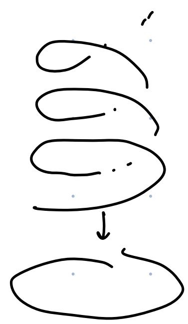
e.g. $S^1 \xrightarrow{z \mapsto z^n} S^1$



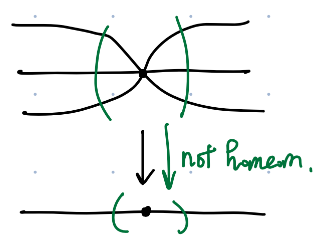
$\mathbb{C}^\times \xrightarrow{z \mapsto z^n} \mathbb{C}^\times$

$\mathbb{R} \longrightarrow S^1$

$\theta \longmapsto e^{i\theta}$.



Rmk: If $f: X \longrightarrow Y$ is a branched cover, then it's not a covering map:



Coro: Let $f: X \longrightarrow Y$ holo. map btw R.S.

If f is injective, then it's a biholomorphism from X to $f(X)$.

pf: Injective $\Rightarrow$ Multiplicity is 1 everywhere in X,

so the inverse $f(X) \xrightarrow{f^{-1}} X$ is holomorphic, since at every pt,

$\exists$ charts so that the map is given by $z \longmapsto z$. $\square$

e.g. (open mapping thm.) $f: \underset{\text{open}}{U \subseteq \mathbb{C}} \longrightarrow \mathbb{C}$ nonconst. holo. Then $f(U)$ is open.

Coro: (open mapping thm). $f: X \longrightarrow Y$ nonconst. holo. map btw R.S.

Then f is open (i.e. sends open subsets of X to open subset of Y).

pf: $\forall\ \mathcal{U} \underset{\text{open}}{\subseteq} X$, write $\mathcal{U} = \bigcup_{a \in \mathcal{U}} \mathcal{U}_a$

open nbhd of a with local normal form.

By the previous thm, $f(\mathcal{U}_a)$ is open in Y. and

$f(\mathcal{U}) = \bigcup_{a \in \mathcal{U}} f(\mathcal{U}_a)$ is open in Y. $\square$

Coro: Let $f: X \longrightarrow Y$ nonconst holo. map. (Suppose Y is connected.)

Suppose X is compact. Then: f is surjective, and Y is compact.

pf: By open mapping thm, $f(X) \underset{\text{open}}{\subseteq} Y$.

X cpt $\Rightarrow$ $f(X)$ cpt. $\Rightarrow$ $f(X)$ closed.

$\Rightarrow$ $f(X) = Y$. since Y is connected. $\square$

Coro: Any holo. function on a cpt. R.S. is constant.

pf: Apply the previous Coro. to $f: X \longrightarrow \mathbb{C}$. $\square$

e.g. Any holo. fcn. on $\mathbb{CP}^1$, or $\mathbb{C}/\Lambda$, is constant.

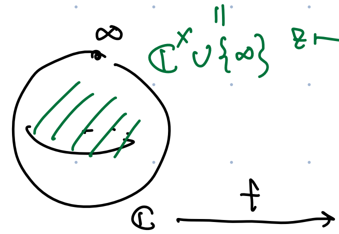
Coro. (Liouville thm.) Any bounded holo. fcn. $f: \mathbb{C} \to \mathbb{C}$ is const.

pf: Claim: We can extend f to a holo. fcn. on $\mathbb{CP}^1$.
(cpt)

Consider the chart at ∞: $U_2 \xrightarrow{\phi_2} \mathbb{C}$

$\quad\; \| \qquad\qquad\qquad$

$\mathbb{C}^\times \cup \{\infty\} \quad z \longmapsto 1/z$.



- $f \circ \phi_2^{-1}$ is holo. & bdd. on $\mathbb{C}^\times$.
- By removable singularity thm, it can be extended to a holo. fcn. on the whole $U_2 \underset{\phi_2}{\xrightarrow{\sim}} \mathbb{C}$.
- Thus, f extends to a holo $\tilde{f}: \mathbb{CP}^1 \longrightarrow \mathbb{C}$.

$\Longrightarrow$ $\tilde{f}$ is constant since $\mathbb{CP}^1$ is compact. $\square$

Rmk: Liouville thm also directly implies that every holo. fcn. on $\mathbb{C}/\Lambda$ is constant.


← compact set.

Prop: $f: X \to Y$ noncont. holo. map btw $\boxed{\text{cpt}}$ R.S. (always assume connected) Then $\exists\, m \geq 1$ s.t. $\forall y \in Y$,

$$\sum_{x \in f^{-1}(y)} m_x(f) = m$$

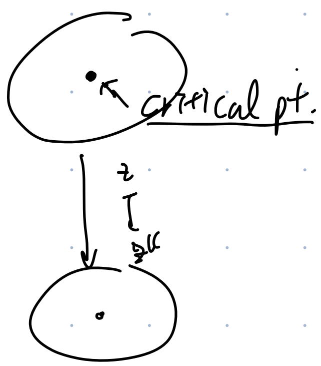
↑ multiplicity of f at x.

Rmk: Such m is called the degree of f ($=: \deg(f)$)
We'll say f is an m-sheeted cover of Y by X.

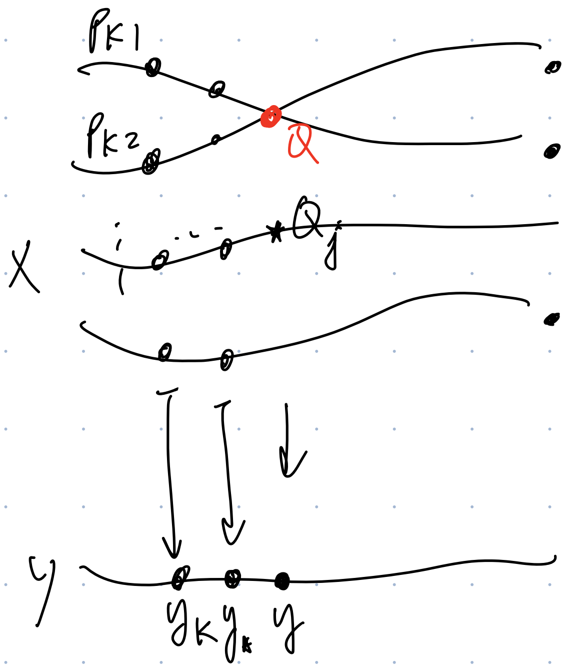
pf: $\forall n \geq 1$, consider

$$Z_n := \left\{ y \in Y \;\middle|\; \sum_{x \in f^{-1}(y)} m_x(f) \geq n \right\}.$$

- $\forall y$, $f^{-1}(y) \subseteq X$ is a finite set. (since X is cpt. and $f^{-1}(y)$ discrete)
- By the local normal form, Z_n is open. (Ex.)
- Claim: Z_n is closed. Let $\lim_{k \to \infty} y_k = y$ where $y_k \in Z_n$.



Note: There are only finitely many critical pts.
⇓ (Crit(f) discrete + X cpt)
We can choose y_k that avoid these critical values (finitely many)
⇓
$f^{-1}(y_k)$ consists of $\geq n$ distinct points

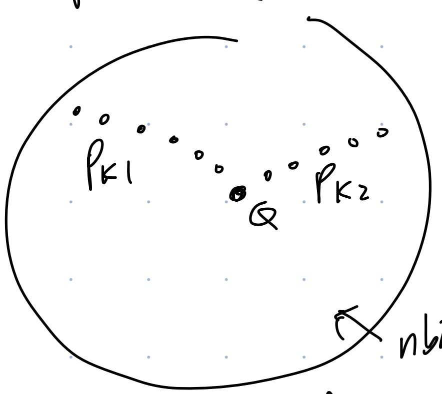


- Say $\{P_{K1}, \dots, P_{Kn}\} \subseteq f^{-1}(y_K)$
- $\forall j \in \{1, \dots, n\}$, $\exists$ subseq. $\{P_{Kj}\}_K$ converges (since X cpt), say to Q_j.
- By continuity of f, we have

$$f(Q_j) = f(\lim_{K\to\infty} P_{Kj}) = \lim_{K\to\infty} f(P_{Kj})$$
$$= \lim_{K\to\infty} y_K = y.$$

- If each Q_j are distinct, then we have immediately that $\sum_{x \in f^{-1}(y)} m_x(f) \geq n$. Thus $y \in Z_n$.

- Suppose there are $\{P_{K1}\}_K$, $\{P_{K2}\}_K$ conv. to the same point Q (as above picture). Then, near Q:



$f(P_{K1}) = f(P_{K2}) = y_K \quad \forall K.$

$\implies m_Q(f) \geq 2.$

nbhd of Q w/ local normal form.

Similarly argument applies in the case when more seq. conv. to a same point. So, we still have: $\sum_{x \in f^{-1}(y)} m_x(f) \geq n$.

$\Rightarrow$ Z_n is either ϕ or the whole Y $\forall n$.

Choose any $y_0 \in Y$, let $m = \sum_{x \in f^{-1}(y_0)} m_x(f)$.

Then $Z_m = Y$.

Since $y_0 \notin Z_{m+1}$, we have $Z_{m+1} = \phi$.

Thus $\sum_{x \in f^{-1}(y)} m_x(f)$ is constant in y. $\square$

e.g:
$$\begin{aligned} \mathbb{CP}^1 &\xrightarrow{\quad f \quad} \mathbb{CP}^1 \\ z &\longmapsto z^n \\ \infty &\longmapsto \infty \end{aligned}$$
degree n.

$\mathrm{Crit}(f) = \{0, \infty\}$. multiplicity n.

$f(0) = 0$, $f(\infty) = \infty$ crit. values.

e.g: Weierstrass $\wp$-fcn: $\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$

defines a holo. map $\wp : \mathbb{C}/\Lambda \longrightarrow \mathbb{CP}^1$ of degree 2.

(will be important for discussions on elliptic curves, modular forms, ...)

<u>Riemann-Hurwitz Thm:</u> $f: X \longrightarrow Y$ surj. holo. map btw (connected) R.S.

Then:

$$\chi(X) = \deg(f) \cdot \chi(Y) - \sum_{x \in \mathrm{Crit}(f)} (m_x(f) - 1)$$

<u>e.g.</u>: $(\mathbb{CP}^1 \xrightarrow{z^n} \mathbb{CP}^1)$:

$$2 = n \cdot 2 - (n-1) - (n-1).$$

<u>e.g.</u>: $(\mathbb{C}/\Lambda \xrightarrow{\wp} \mathbb{CP}^1)$:

$$0 = 2 \cdot 2 - 1 - 1 - 1 - 1.$$

<u>A topological proof of R-H:</u>

- Choose a triangulation of Y s.t. every Crit. value is a vertex of the triangulation.
- Can pullback the triangulation via f, to a triangulation of X.
- $E_X = n E_Y$, $F_X = n F_Y$. $(n := \deg(f))$
- $V_X = n V_Y - \sum_{x \in \mathrm{Crit}(f)} (m_x(f) - 1)$

$$\Rightarrow \chi(X) = V_X - E_X + F_X$$

$$= n\chi(Y) - \sum_{x \in \mathrm{Crit}(f)} (m_x(f) - 1). \quad \square$$

Def: $Y \underset{\text{open}}{\subseteq} X$: R.S. A <u>meromorphic fcn</u> on Y is:

- a holo. fcn. $Y' \longrightarrow \mathbb{C}$ s.t.
- $Y \setminus Y'$ is a set of isolated points, (<u>poles</u>)
- $\forall a \in Y \setminus Y'$, $\lim_{x \to a} |f(x)| = \infty$.

Thm: Let X : R.S., f: mero. fcn. on X.

For each pole $a \in X$, let $f(a) := \infty$.

Then, $f: X \longrightarrow \mathbb{CP}^1$ is holo.

Conversely, $\forall f: X \longrightarrow \mathbb{CP}^1$ holo,

either $f \equiv \{\infty\}$ or $f^{-1}(\infty)$ consists of isolated pts

and $f: X \setminus f^{-1}(\infty) \longrightarrow \mathbb{C}$ holo.

Rmk: Thus, we can identify mero. fcn. on X with holo. maps $f: X \longrightarrow \mathbb{CP}^1$.

pf: The extension $f: X \to \mathbb{CP}^1$ is clearly continuous,

It's holo. by removable sing. thm.

The converse follows from the identity thm. $\square$

NB: $\mathcal{M}(X) := \{$mero. fcn. on $X\}$ is a field.

(f has only isolated zeros. (therefore admits multiplicative inverse), unless $f \equiv 0$).

called the function field of X.

Thm: $\mathcal{M}(\mathbb{CP}^1) = \{$rational functions $\frac{P(x)}{Q(x)}\} = \mathbb{C}(x)$

pf: Clearly we have "$\supseteq$".

Suppose $f \in \mathcal{M}(\mathbb{CP}^1)$.

- May assume ∞ is not a pole of f (i.e. $f(\infty) \neq \infty$), otherwise, one can replace f by $1/f$.

- Let $a_1, \dots, a_n \in \mathbb{C}$ be the poles of f. (finitely many, since $\mathbb{CP}^1$ compact).

- For each a_i, let $\underbrace{\frac{*}{(z-a_i)^{k_i}} + \dots + \frac{*}{z-a_i}}_{=: h_i(z)}$ be the principal part of f at a_i.

- Then $f - (h_1 + \dots + h_n)$ is holo. on $\mathbb{CP}^1$. $\Rightarrow$ const.

$\Rightarrow$ f is rational. $\square$