Def: A <u>Riemann surface</u> (or <u>R.S.</u>) is the data of:

- a Hausdorff topological space X, with:
- an <u>atlas</u> $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ where:
  - each $U_\alpha \subseteq X$ is open
  - $\bigcup_{\alpha \in A} U_\alpha = X$. ($(U_\alpha, \phi_\alpha)$ is called a <u>chart</u>)
  - $\phi_\alpha : U_\alpha \longrightarrow \mathbb{C}$ is homeomorphism onto its image.
  - the <u>transition map</u> $\phi_\beta \circ \phi_\alpha^{-1} : \underset{\mathbb{C}}{\underset{\cap}{\phi_\alpha(U_\alpha \cap U_\beta)}} \xrightarrow{\sim} \underset{\mathbb{C}}{\underset{\cap}{\phi_\beta(U_\alpha \cap U_\beta)}}$ is <u>holomorphic</u> $\forall \alpha, \beta \in A$.

Note: We say two atlases on X define the same R.S. if their union is also an atlas. (i.e. they're holomorphically compatible).

Rmk:

- If $p \in U_\alpha$, we can think of $\phi_\alpha : U_\alpha \to \mathbb{C}$ as a local holomorphic coordinate z near the point p,
- Note that the local coordinate z at a point $p \in X$ is <u>NOT</u> unique. Suppose $p \in U_\beta$ in another chart $\rightsquigarrow$ different local coordinate w. where z and w are related by a holomorphic change of coordinate $\phi_\beta \circ \phi_\alpha^{-1}$. So, we'll be mainly interested in notions that are <u>independent</u> of the choice of particular local coordinate.
- If we replace the holomorphicity condition on the transition maps by: requiring them to be <u>smooth</u> $\rightsquigarrow$ smooth (real) surface
  or <u>smooth with positive Jacobian</u> $\rightsquigarrow$ smooth oriented surface
  or no condition $\rightsquigarrow$ topological surface.

Def: Let $(X, \{U_\alpha, \phi_\alpha\})$, $(Y, \{V_\beta, \psi_\beta\})$ be R.S.

A <u>holomorphic map</u> $f: X \longrightarrow Y$ is a continuous map s.t. $\forall \alpha, \beta$,

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \longrightarrow \psi_\beta(V_\beta) \text{ is holomorphic}$$

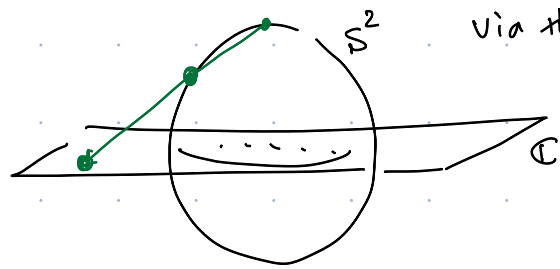
- We say two R.S. X and Y are <u>equivalent</u> if $\exists f: X \longrightarrow Y$ holomorphic bijection with holomorphic inverse.

<u>e.g.</u> $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathbb{H} = \{w \in \mathbb{C} \mid \mathrm{Im}(w) > 0\}$ both R.S.

They're equivalent, through $z = \dfrac{w-i}{w+i}$

<u>e.g.</u> <u>Riemann sphere</u> ($\hat{\mathbb{C}}$, $\mathbb{C} \cup \{\infty\}$, $\mathbb{P}^1$, $\mathbb{CP}^1$, complex projective line)

- <u>Topology</u>: open subsets are: either open subsets of $\mathbb{C}$, or is of the form $U \cup \{\infty\}$ where $U = \mathbb{C} \setminus K$ for some $\underset{\text{compact}}{K \subseteq \mathbb{C}}$.
- It's a Hausdorff top. space, homeomorphic to S^2. via the stereographic projection.



- We can equip $\mathbb{C} \cup \{\infty\}$ with the following complex structure:
  - $U_1 = \mathbb{C} \xrightarrow[\cong]{\phi_1 = id} \mathbb{C}$
  - $U_2 = \mathbb{C}^\times \cup \{\infty\} \xrightarrow[\cong]{\phi_2 : z \mapsto \frac{1}{z}} \mathbb{C}$

  are homeomorphisms.

Transition map $\underset{\mathbb{C}^\times}{\overset{}{\phi_1(U_1 \cap U_2)}} \xrightarrow[z \mapsto \frac{1}{z}]{\phi_2 \circ \phi_1^{-1}} \phi_2(U_1 \cap U_2)$ is holomorphic e.g. (Algebraic curves):

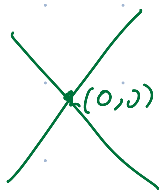
Let $P(z,w) \in \mathbb{C}[z,w]$.

Define $X := \{(z,w) \in \mathbb{C}^2 \mid P(z,w) = 0\}$.

Assume that $\forall (z_0,w_0) \in X$, $\left(\frac{\partial P}{\partial z}(z_0,w_0), \frac{\partial P}{\partial w}(z_0,w_0)\right) \neq (0,0)$.

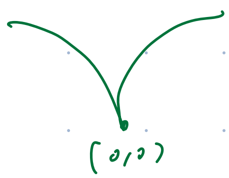
Then, we'll show that X is naturally a R.S.

(Non-e.g. $z^2 - w^2$:



nodal singularity

$z^2 - w^3$



cuspidal singularity)

Thm: Suppose $(z_0,w_0) \in \overset{\{f(z,w)=0\}}{\overset{\shortparallel}{X}}$ with $\frac{\partial f}{\partial w}(z_0,w_0) \neq 0$. Then,

$\exists$ a disc $\mathbb{D}_1$ centered at z_0 in $\mathbb{C}$,

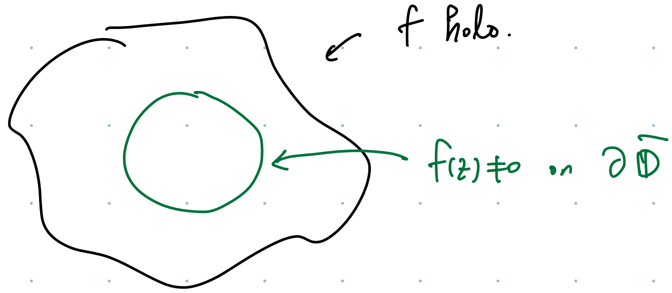
a disc $\mathbb{D}_2$ centered at w_0 in $\mathbb{C}$,

a holomorphic map $\phi: \mathbb{D}_1 \longrightarrow \mathbb{D}_2$, $\phi(z_0) = w_0$.

s.t. $X \cap \mathbb{D}_1 \times \mathbb{D}_2 = \{(z, \phi(z)) \mid z \in \mathbb{D}_1\}$

Note: This holds for f holomorphic. not necessarily polynomial.

pf: Recall the <u>argument principle</u>:

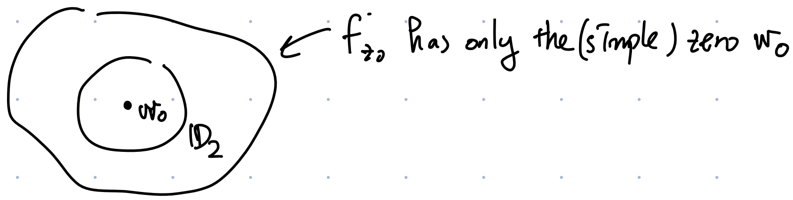


- $\displaystyle \frac{1}{2\pi i}\int_{\partial \bar{\mathbb{D}}} \frac{f'(w)}{f(w)}\,dw = \#$ zeros of f in $\mathbb{D}$.
- $\displaystyle \frac{1}{2\pi i}\int_{\partial \bar{\mathbb{D}}} \frac{w f'(w)}{f(w)}\,dw = \sum_{\substack{z\in\mathbb{D},\\ f(z)=0.}} z\cdot \mathrm{mult}(f;z)$ sum of the zeros.

Consider $f_{z_0}(w) := P(z_0, w)$. Then $f_{z_0}(w_0)=0$, $f'_{z_0}(w_0)\neq 0$

so $\mathrm{mult}(f_{z_0}; w_0) = 1$.

By isolation of zeros, $\exists\, \mathbb{D}_2$ centered at w_0 s.t.



$$\Rightarrow \frac{1}{2\pi i}\int_{\partial \mathbb{D}_2} \frac{f'_{z_0}}{f_{z_0}} = 1.$$

Since $f_{z_0} \neq 0$ on $\partial\mathbb{D}_2$, $\exists\, \delta>0$ s.t. $|f_{z_0}| > 2\delta > 0$ on $\partial\mathbb{D}_2$.

$\Rightarrow \exists$ a disc $\mathbb{D}_1$ centered at z_0 s.t. $\forall z\in\mathbb{D}_1$, $|f_z| > \delta$ on $\partial\mathbb{D}_2$

$\Rightarrow \quad \frac{1}{2\pi i}\int_{\partial \mathbb{D}_2} \frac{f_z'}{f_z} = 1 \quad$ for $z \in \mathbb{D}_1$.

by continuity, and the fact that the integral $\in \mathbb{Z}$.

$\Rightarrow \quad \forall\, z \in \mathbb{D}_1, \quad \exists!\, w \in \mathbb{D}_2$ s.t. $\underset{f(z,w)}{\underset{\parallel}{f_z(w)}} = 0$, and it's a simple zero of f_z.

In fact, $\quad w = \frac{1}{2\pi i}\int_{\partial \mathbb{D}_2} \frac{t f_z'(t)}{f_z(t)}\, dt$

It remains to show that the map:

$$\begin{aligned} \mathbb{D}_1 &\longrightarrow \mathbb{D}_2 \\ z &\longmapsto \frac{1}{2\pi i}\int_{\partial \mathbb{D}_2} \frac{t f_z'(t)}{f_z(t)}\, dt \\ &\qquad \parallel \\ &\quad \frac{1}{2\pi i}\int_{\partial \mathbb{D}_2} \frac{t}{f(z,t)} \frac{\partial f}{\partial t}(z,t)\, dt \end{aligned}$$

is holomorphic

which is easy to see it's holomorphic. $\square$

Returning to the example of alg. curves: $\forall (z_0, w_0) \in X = \{p(z,w) = 0\}$, we assumed that $\left(\frac{\partial P}{\partial z}(z_0, w_0), \frac{\partial P}{\partial w}(z_0, w_0)\right) \neq (0,0)$.

WLOG, say $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$. By the implicit fcn thm above,

$\exists$ $\mathbb{D}_1$ centered at z_0, $\mathbb{D}_2$ centered at w_0, $\phi: \mathbb{D}_1 \longrightarrow \mathbb{D}_2$, $z_0 \longmapsto w_0$

s.t. $X \cap \mathbb{D}_1 \times \mathbb{D}_2 = \{(z_0, \phi(z_0)) \mid z_0 \in \mathbb{D}_1\}$.

Then,
$$\begin{aligned} \mathcal{U} := X \cap \mathbb{D}_1 \times \mathbb{D}_2 &\xrightarrow{\psi} \mathbb{D}_1 \subseteq \mathbb{C} \\ (z_0, \phi(z_0)) &\longmapsto z_0 \end{aligned}$$
gives a chart.

Similarly, if $\frac{\partial P}{\partial z}(z_0, w_0) \neq 0$, then the projection to the 2nd factor gives a chart.

<u>Claim</u>. These charts are holomorphically compatible.

<u>pf</u>:
- If both charts are of the same type (i.e. projection to the same factor) then the transition map is just the identity map.
- For charts of different type, the transition map is given by:

$$\begin{aligned} \mathbb{D}_1 &\xrightarrow{\psi^{-1}} \mathcal{U} \xrightarrow{\pi_2} \mathbb{D}_2 \\ z_0 &\longmapsto (z_0, \phi(z_0)) \longmapsto \phi(z_0). \end{aligned}$$

which is holo. since ϕ is holo. $\square$

<u>Rmk</u>: The same argument works for P holo., not necessarily polynomial.

Projective algebraic plane curve

Recall: $\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^\times$ complex projective space.

elements usually denoted by $[z_0, \dots, z_n]$.

- $\mathbb{C}P^n = U_0 \cup \dots \cup U_n$, where U_i consists of points where $z_i \neq 0$,
  so $U_i = \{[z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n] \mid z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n \in \mathbb{C}\} \cong \mathbb{C}^n$
  This gives a complex manifold structure on $\mathbb{C}P^n$, where each $U_i \xrightarrow{\sim} \mathbb{C}^n$ gives a chart.

- $p \in \mathbb{C}[z_1, z_2]$
  say $p = \sum a_{ij} z_1^i z_2^j$. $\overset{\text{homogenization}}{\rightsquigarrow}$ $P(z_0, z_1, z_2) := \sum a_{ij} z_0^{d-i-j} z_1^i z_2^j$
  with $\deg(p) = d$. homog. poly. of deg d.

  e.g. $z_1^2 + z_2^3 + 1 \rightsquigarrow z_0 z_1^2 + z_2^3 + z_0^3$.

- $X = \{(z_1, z_2) \in \mathbb{C}^2 \mid p(z_1, z_2) = 0\} \subseteq \mathbb{C}^2$

  $\cap$ $(z_1, z_2) \mapsto [1, z_1, z_2]$

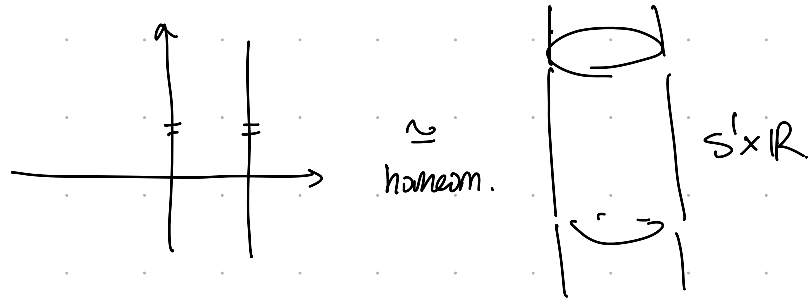
  $\overline{X} = \{[z_0, z_1, z_2] \in \mathbb{C}P^2 \mid P(z_0, z_1, z_2) = 0\} \subseteq \mathbb{C}P^2$

- $\mathbb{C}P^2$ is compact, and $\overline{X} \underset{\text{closed}}{\subseteq} \mathbb{C}P^2 \Rightarrow \overline{X}$ is compact.

Ex: Show that $\overline{X}$ is a R.S.

Quotients:

e.g. $\mathbb{C}/\mathbb{Z}$.



- $\forall z \in \mathbb{C}$, consider the disc $\mathbb{D}_z(1/4)$. Then $\forall z_1, z_2 \in \mathbb{D}_z(1/4)$

  if $z_1 \sim z_2$, i.e. $z_1 = z_2 + n$ for some $n \in \mathbb{Z}$,

  then $z_1 = z_2$ and $n = 0$.

$$\Rightarrow \quad \begin{array}{ccc} \pi : \mathbb{C} & \twoheadrightarrow & \mathbb{C}/\mathbb{Z} \\ \cup\!| & & \cup\!| \\ \mathbb{D}_z(1/4) & \xrightarrow{\sim} & \pi(\mathbb{D}_z(1/4)) \end{array}$$

$\Rightarrow$ We can use this to construct a chart about $\pi(z) \in \mathbb{C}/\mathbb{Z}$.

- The transition maps btw charts are of the form $z \longmapsto z + n$

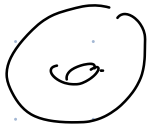
  which are clearly holo.

  This gives $\mathbb{C}/\mathbb{Z}$ a R.S. structure.

- In fact,
$$\begin{array}{ccc} \mathbb{C}/\mathbb{Z} & \xrightarrow{\sim} & \mathbb{C}^\times \\ z & \longmapsto & e^{2\pi i z}. \end{array}$$

e.g. Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$.

We can similarly define a R.S. str. on $\mathbb{C}/\Lambda$, where we choose the radius r s.t. $2r < \min_{\lambda \in \Lambda} |\lambda|$

$\rightsquigarrow$ Complex torus  $\simeq S^1 \times S^1$.

which is a compact R.S.

In general, suppose a group Γ acts on a R.S. X by holo. automorphisms and

- $\forall p \in X$, $\exists\, p \in \mathcal{U} \underset{\text{open}}{\subseteq} X$ s.t. $\forall q_1, q_2 \in \mathcal{U}$ " $q_1 = \gamma(q_2)$ for some $\gamma \in \Gamma \implies q_1 = q_2$ and $\gamma = 1$ ".
- $\forall \bar{p}_1 \neq \bar{p}_2 \in X/\Gamma$, $\exists\, p_1 \in \mathcal{U}_1, p_2 \in \mathcal{U}_2$ s.t. $\Gamma\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$.

Then the same construction endows a R.S. str. on X/Γ.

Let's look at the holo. auto. of some basic R.S. (We'll discuss in more details later on.)

e.g. $\mathrm{Aut}(\mathbb{CP}^1) = \left\{ z \mapsto \frac{az+b}{cz+d} \text{, where } \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \right\} \cong PSL_2(\mathbb{C})$.

e.g. $\mathrm{Aut}(\mathbb{C}) = \{ z \mapsto az+b \text{, where } a \neq 0 \} \cong \mathbb{C}^\times \times \mathbb{C}$.

e.g. $\mathrm{Aut}(\mathbb{H}) \cong PSL_2(\mathbb{R}) = \{ \mu \in PSL_2(\mathbb{C}) \mid \mu(\mathbb{H}) = \mathbb{H} \}$.