

RIEMANN SURFACES, HOMEWORK 4
DUE JUNE 9 AT 9:55AM

Some ground rules:

- **You can submit your solutions in class or on the eLearning system (<https://elearning.fudan.edu.cn>).**
- **Late Submission Policy:** If you submit the homework $N \leq 10$ days late, your score will be multiplied by $(1 - \frac{N}{10})$. Submissions more than 10 days late will receive a score of zero automatically.
- You may use English, Chinese, or a combination of both in your solutions.
- Write your argument as clearly as possible, and ensure your final submission is legible.
- Feel free to use results that are proved in class. If you wish to use a result from outside of class, you must provide a complete proof of it before using it.
- Collaboration on assignments is encouraged. If you work with others, you must **acknowledge your collaborators** in your solutions. However, you are expected to **write your own solutions independently**.

Problems:

(1) Let X be a compact Riemann surface of genus two. Suppose ω_1, ω_2 form a basis of $H^0(X, \mathcal{O}^1)$ and define $f \in \mathcal{M}(X)$ by $\omega_1 = f\omega_2$. Prove that $f: X \rightarrow \mathbb{P}^1$ is a two-sheeted branched cover.

(2) Let X be a compact Riemann surface, and $Y \subseteq X$ be an open subset such that $X \setminus Y$ has non-empty interior. Let D be a divisor on X . Prove that there exists a function $f \in \mathcal{M}(X)$ such that $\text{ord}_y(f) = D(y)$ for all $y \in Y$.

(Hint: Find a divisor D' with support in $X \setminus Y$ such that $D + D'$ is a principal divisor.)

(3) Let D be a divisor on a compact Riemann surface X of genus g , such that $\deg(D) = 2g - 2$ and $h^0(\mathcal{O}_D) = g$. Prove that D is a canonical divisor.

(4) Let X be a compact Riemann surface of genus one.

- (a) Prove that X is isomorphic to a complex torus \mathbb{C}/L .
- (b) Identifying X with \mathbb{C}/L (and choose the base point $p_0 = 0 \bmod L$), prove that the Abel–Jacobi map $X \rightarrow \text{Jac}(X)$ is an isomorphism of groups by explicitly computing the integrals.

(5) Let X be a compact Riemann surface of genus at least one. Prove that the Abel–Jacobi map $X \rightarrow \text{Jac}(X)$ is injective.