RIEMANN SURFACES, HOMEWORK 3
DUE MAY 19 AT 9:55AM

Some ground rules:

- **You can submit your solutions in class or on the eLearning system (*https://elearning.fudan.edu.cn*).**
- **Late Submission Policy:** If you submit the homework $N \leq 10$ days late, your score will be multiplied by $(1 - \frac{N}{10})$. Submissions more than 10 days late will receive a score of zero automatically.
- You may use English, Chinese, or a combination of both in your solutions.
- Write your argument as clearly as possible, and ensure your final submission is legible.
- Feel free to use results that are proved in class. If you wish to use a result from outside of class, you must provide a complete proof of it before using it.
- Collaboration on assignments is encouraged. If you work with others, you must **acknowledge your collaborators** in your solutions. However, you are expected to **write your own solutions independently**.

Problems:

(1) Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be sheaves on a topological space X. Suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$$

is an exact sequence of sheaves (i.e., exact on every stalk). Prove that

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$$

is an exact sequence of abelian groups for any open set $U \subseteq X$.

(2) Prove that the following sequence of sheaves on a Riemann surface X

$$0 \to \underline{\mathbb{C}}^\times \to \mathcal{O}^\times \xrightarrow{\mathrm{d}\log} \mathcal{O}^1 \to 0$$

is exact, where $\underline{\mathbb{C}}^\times$ is the locally constant sheaf, $\mathcal{O}^1$ is the sheaf of holomorphic 1-forms, and $(\mathrm{d}\log)f = \frac{\mathrm{d}f}{f}$.

(3) Let $\mathcal{U}$ be an open cover of a space X, and $\mathcal{F}$ a sheaf on X. Prove that

$$\cdots \to C^{p-1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^{p-1}} C^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^p} C^{p+1}(\mathcal{U}, \mathcal{F}) \to \cdots$$

is a complex, meaning that $\delta^p \circ \delta^{p-1} = 0$ for all $p \geq 1$.

(4) Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ be open covers of X, where $\mathcal{V}$ is a refinement of $\mathcal{U}$. Suppose that $\tau_1 \colon J \to I$ and $\tau_2 \colon J \to I$ both satisfy

$$V_j \subseteq U_{\tau_1(j)} \quad \text{and} \quad V_j \subseteq U_{\tau_2(j)} \quad \text{for all } j \in J.$$

Prove that the induced homomorphisms on Čech cohomology

$$\tau_i^{*p} \colon H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{V}, \mathcal{F})$$

coincide, i.e., $\tau_1^{*p} = \tau_2^{*p}$. (Hint: Complete the proof that we discussed in class.)

(5) Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be sheaves on X. Suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of sheaves. Prove that there is an associated long exact sequence in cohomology:

$$\begin{aligned} 0 &\to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \\ &\to H^1(X, \mathcal{F}_1) \to H^1(X, \mathcal{F}_2) \to H^1(X, \mathcal{F}_3) \\ &\to H^2(X, \mathcal{F}_1) \to \cdots \end{aligned}$$

(6) Let $\mathbb{D} \subseteq \mathbb{C}$ denote the open unit disc. Let $\mathcal{H}$ be the sheaf of complex-valued harmonic functions on $\mathbb{D}$. Prove that $H^1(\mathbb{D}, \mathcal{H}) = 0$.

(7) Consider the open covering $\mathcal{U} = \{\mathbb{P}^1 \setminus \{\infty\}, \mathbb{P}^1 \setminus \{0\}\}$ of $\mathbb{P}^1$. Let $U_1 = \mathbb{P}^1 \setminus \{\infty\}$ and $U_2 = \mathbb{P}^1 \setminus \{0\}$.

(a) Prove that $\mathcal{U}$ is a Leray covering for the sheaf of holomorphic 1-forms $\mathcal{O}^1$ on $\mathbb{P}^1$.

(b) Prove that $H^1(\mathbb{P}^1, \mathcal{O}^1) \cong H^1(\mathcal{U}, \mathcal{O}^1) \cong \mathbb{C}$, and that the class of

$$\frac{\mathrm{d}z}{z} \in \mathcal{O}^1(U_1 \cap U_2) \cong Z^1(\mathcal{U}, \mathcal{O}^1)$$

forms a basis for $H^1(\mathbb{P}^1, \mathcal{O}^1)$.

(8) Let D be a divisor on $\mathbb{P}^1$. Prove that:

(a) $h^0(\mathbb{P}^1, \mathcal{O}_D) = \max\{0, 1 + \deg(D)\}$.

(b) $h^1(\mathbb{P}^1, \mathcal{O}_D) = \max\{0, -1 - \deg(D)\}$.

(9) Let $X = \mathbb{C}/\Lambda$ be a torus, and $x_0 \in X$ a point. Prove that

$$h^0(X, \mathcal{O}_{nx_0}) = \begin{cases} 0 & \text{for } n < 0, \\ 1 & \text{for } n = 0, \\ n & \text{for } n > 0. \end{cases}$$

(Hint: Use the Weierstrass $\wp$-function).

(10) Prove that the map

$$[1 : \wp : \wp'] \colon \mathbb{C}/\Lambda \to \mathbb{P}^2$$

is an embedding, where $\wp$ denotes the Weierstrass $\wp$-function associated to the lattice Λ.