RIEMANN SURFACES, HOMEWORK 2
DUE APRIL 21 AT 9:55AM

Some ground rules:

- **You can submit your solutions in class or on the eLearning system (*https://elearning.fudan.edu.cn*).**
- **Late Submission Policy:** If you submit the homework $N \leq 10$ days late, your score will be multiplied by $(1 - \frac{N}{10})$. Submissions more than 10 days late will receive a score of zero automatically.
- You may use English, Chinese, or a combination of both in your solutions.
- Write your argument as clearly as possible, and ensure your final submission is legible.
- Feel free to use results that are proved in class. If you wish to use a result from outside of class, you must provide a complete proof of it before using it.
- Collaboration on assignments is encouraged. If you work with others, you must **acknowledge your collaborators** in your solutions. However, you are expected to **write your own solutions independently**.

Problems:

(1) Let X be a compact Riemann surface with $g(X) \geq 1$.

(a) Construct a non-constant smooth map $\mathbb{P}^1 \to X$.

(b) Prove that there is no non-constant holomorphic map $\mathbb{P}^1 \to X$.

(2) Consider the universal covering map

$$\exp\colon \mathbb{C} \to \mathbb{C}^\times.$$

Compute

$$\exp^*\left(\frac{\mathrm{d}z}{z}\right).$$

(3) Consider

$$\tan = \frac{\sin}{\cos}\colon \mathbb{C} \to \mathbb{P}^1.$$

(a) Prove that tan is a local homeomorphism, and the image is $\tan(\mathbb{C}) = \mathbb{P}^1 \backslash \{\pm i\}$.

(b) Prove that

$$\tan\colon \mathbb{C} \to \mathbb{P}^1 \backslash \{\pm i\}$$

is a covering map.

(c) Prove that the holomorphic 1-form

$$\frac{\mathrm{d}z}{1+z^2}$$

on $\mathbb{C}\backslash\{\pm i\}$ can be extended to a holomorphic 1-form ω on $\mathbb{P}^1 \backslash \{\pm i\}$.

(d) Compute $\tan^*(\omega)$.

(4) Let $f\colon X \to Y$ be a holomorphic map between Riemann surfaces, and let $k \geq 1$ be the multiplicity of f at $x_0 \in X$. Let ω be a meromorphic 1-form on Y with a pole at $f(x_0)$. Prove that

$$\operatorname{res}_{x_0}(f^*\omega) = k \cdot \operatorname{res}_{f(x_0)}(\omega).$$

(5) Let $\Lambda \subseteq \mathbb{C}$ be a lattice, and $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ the quotient map.

- (a) Prove that $\mathrm{d}z$ defines a global holomorphic 1-form on $\mathbb{C}/\Lambda$, with no zeros or poles.
- (b) Prove that $\mathrm{d}\bar{z}$ defines a global smooth 1-form on $\mathbb{C}/\Lambda$.
- (c) Prove that every holomorphic 1-form on $\mathbb{C}/\Lambda$ is a constant multiple of $\mathrm{d}z$.
- (d) Prove that $\mathrm{d}z \wedge \mathrm{d}\bar{z}$ defines a global smooth 2-form on $\mathbb{C}/\Lambda$.

(6) Let $X = \mathbb{C}/\Lambda$ be a complex torus. Given a group homomorphism

$$F\colon \pi_1(X) \to \mathbb{C}.$$

Prove that there exists a closed 1-form ω on X such that

$$\int_{[\gamma]} \omega = F([\gamma]) \qquad \text{for any } [\gamma] \in \pi_1(X).$$

(7) Let $X \subseteq \mathbb{C}^2$ be a smooth plane curve defined as the zero set of a polynomial $p(x, y) = 0$.

- (a) Prove that $\mathrm{d}x$ and $\mathrm{d}y$ define holomorphic 1-forms on X, so do $q(x, y)\,\mathrm{d}x$ and $q(x, y)\,\mathrm{d}y$ for any polynomial q.
- (b) Prove that

$$\left(\frac{\partial p}{\partial x}\right) \mathrm{d}x = -\left(\frac{\partial p}{\partial y}\right) \mathrm{d}y$$

as holomorphic 1-forms on X.

(8) Let $p(x)$ be a polynomial of degree $2g + 1$ or $2g + 2$ with no repeated roots.

- (a) Prove that $y^2 = p(x)$ defines a smooth plane curve.
- (b) Prove that one can compactify it to obtain a compact Riemann surface X, which admits a degree 2 map $\pi\colon X \to \mathbb{P}^1$ given by the x-coordinate. *(Such a Riemann surface X is called a hyperelliptic curve.)*
- (c) Prove that the critical values of π are the roots of $p(x)$, together with $\infty \in \mathbb{P}^1$ when $\deg p(x) = 2g + 1$.
- (d) Use the Riemann–Hurwitz formula to show that $g(X) = g$.

(9) Continue the same setting as the previous problem.

- (a) Prove that the 1-forms

$$\frac{\mathrm{d}x}{y}, \ \frac{x\,\mathrm{d}x}{y}, \ \ldots, \ \frac{x^{g-1}\,\mathrm{d}x}{y}$$

are holomorphic 1-forms on X.

(b) Consider the meromorphic 1-form $\frac{\mathrm{d}x}{p(x)}$ on $\mathbb{P}^1$. Let

$$\omega = \pi^* \left(\frac{\mathrm{d}x}{p(x)} \right)$$

be its pullback. Compute the orders of ω at all of its zeros and poles.

(10) A complex torus X can be represented as a smooth cubic curve in $\mathbb{P}^2$, say $y^2 z = x^3 - g_2 x^2 z - g_3 z^3$, where the cubic polynomial $x^3 - g_2 x - g_3$ has no repeated roots. It seems that

$$\frac{\mathrm{d}x}{y}, \ \frac{x\,\mathrm{d}x}{y}, \ \frac{x^2\,\mathrm{d}x}{y}$$

are three linearly independent holomorphic 1-forms. However, this would contradict the fact that the space of holomorphic 1-forms on a complex torus is 1-dimensional. What is the resolution of this paradox?