RIEMANN SURFACES, HOMEWORK 1
DUE MARCH 31 AT 9:55AM

Some ground rules:

- **You can submit your solutions in class or on the eLearning system (*https://elearning.fudan.edu.cn*).**
- **Late Submission Policy:** If you submit the homework $N \leq 10$ days late, your score will be multiplied by $(1 - \frac{N}{10})$. Submissions more than 10 days late will receive a score of zero automatically.
- You may use English, Chinese, or a combination of both in your solutions.
- Write your argument as clearly as possible, and ensure your final submission is legible.
- Feel free to use results that are proved in class. If you wish to use a result from outside of class, you must provide a complete proof of it before using it.
- Collaboration on assignments is encouraged. If you work with others, you must **acknowledge your collaborators** in your solutions. However, you are expected to **write your own solutions independently**.

Problems:

(1) Let $P \in \mathbb{C}[z_0, z_1, z_2]$ be a non-constant homogeneous polynomial, and let

$$X = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 : P(z_0, z_1, z_2) = 0\} \subseteq \mathbb{CP}^2.$$

Assume that for every point $[z_0, z_1, z_2] \in X$, we have

$$\left(\frac{\partial P}{\partial z_0}(z_0, z_1, z_2), \frac{\partial P}{\partial z_1}(z_0, z_1, z_2), \frac{\partial P}{\partial z_2}(z_0, z_1, z_2)\right) \neq (0, 0, 0).$$

Prove that X is a Riemann surface.

(2) Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Prove that the set of critical points is a discrete subset of X.

(3) In class, we defined the *multiplicity* of a non-constant holomorphic map $f : X \to Y$ between Riemann surfaces at a point $a \in X$ as follows: We proved that there exist charts (U, ϕ) (where $a \in U \subseteq X$) and (V, ψ) (where $f(a) \in V \subseteq Y$) such that $f(U) \subseteq V$ and the composition

$$\phi(U) \xrightarrow{\psi \circ f \circ \phi^{-1}} \psi(V)$$

is given by $z \mapsto z^k$; we then defined the multiplicity $m_a(f) \coloneqq k \geq 1$.

Prove that this definition is *independent* of the choice of the charts. In other words, suppose there exist other charts $(U,' \phi')$ (where $a \in U' \subseteq X$) and (V', ψ') (where $f(a) \in V' \subseteq Y$) such that $f(U') \subseteq V'$ and the composition

$$\phi(U') \xrightarrow{\psi' \circ f \circ \phi'^{-1}} \psi'(V)$$

is given by $z \mapsto z^\ell$. Prove that $k = \ell$.

(4) Let X be a connected Riemann surface, and $f\colon X \to \mathbb{C}$ be a holomorphic function. Suppose there exists a point $x_0 \in X$ such that

$$|f(x)| \leq |f(x_0)| \quad \text{for all } x \in X.$$

Prove that f is a constant function.

(5) Prove that any holomorphic automorphism $f\colon \mathbb{C} \to \mathbb{C}$ must be of the form $f(z) = az + b$ for some $(a, b) \in \mathbb{C}^\times \times \mathbb{C}$.

(6) Consider the projective plane curve of degree d defined by

$$X = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 : z_0^d + z_1^d + z_2^d = 0\} \subseteq \mathbb{CP}^2.$$

(i) Show that X is a Riemann surface (i.e. it has no singular points, cf. the condition in Problem 1).
(ii) Show that the map $\pi\colon X \to \mathbb{P}^1$ defined by $[z_0, z_1, z_2] \mapsto [z_0, z_1]$ is holomorphic, and compute its degree.
(iii) Find all critical points and critical values of π.
(iv) Apply the Riemann–Hurwitz formula to the map π, and deduce that

$$g(X) = \frac{(d-1)(d-2)}{2}.$$

(7) Suppose γ_1 (resp. γ_2) and σ_1 (resp. σ_2) are homotopic paths in X with $\gamma_1(1) = \sigma_1(1) = \gamma_2(0) = \sigma_2(0)$. Prove that:

(i) $\gamma_1 \cdot \gamma_2$ is homotopic to $\sigma_1 \cdot \sigma_2$.
(ii) γ_1^{-1} is homotopic to σ_1^{-1}.

(8) Let $\gamma_1, \gamma_2, \gamma_3$ be paths in X such that $\gamma_1(1) = \gamma_2(0)$ and $\gamma_2(1) = \gamma_3(0)$. Prove that:

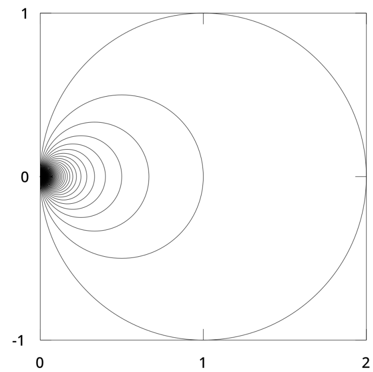
(i) $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$ is homotopic to $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3$.
(ii) $\gamma_1 \cdot \gamma_1^{-1}$ is homotopic to a constant path.

(9) In class, we constructed a map from $\pi_1(S^1)$ to $\mathbb{Z}$, and showed that it is well-defined. Complete the proof that it is a group isomorphism (i.e. prove that it is a group homomorphism; it is injective and surjective.)

(10) Define the *infinite circle space* $A \subseteq \mathbb{R}^2$ to be the union

$$A := \bigcup_{n=1}^{\infty} \left\{ (x, y) \in \mathbb{R}^2 : \left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2 \right\},$$

equipping with the subspace topology from $\mathbb{R}^2$.



Define B to be the *wedge sum of countably infinite many circles*

$$B := \bigvee_{n=1}^{\infty} S^1$$

which is the disjoint union of countably many circles glued at a single point, equipped with the topology where a subset is open if and only if its intersection with each S^1 is open.

(i) Construct an explicit continuous bijection $f\colon B \to A$ that maps the wedge point to the origin $(0,0) \in A$, and show that f is not a homeomorphism.

(ii) Show that each loop in B can only go around finitely many circles, while loops in A can wind around infinitely many circles.