MATHEMATICS FROM EXAMPLES, SPRING 2023

INSTRUCTOR: YU-WEI FAN

Course Description. In mathematics, examples are analogous to phenomena in physics. They are integral to the historical progression of mathematical thought, driving the development of profound concepts and methodologies. Many significant theorems in modern mathematics emerge from the study and analysis of basic examples. This course aims to elucidate abstract mathematical concepts by presenting intriguing examples, thereby fostering motivation and intuition among students.

Contents

1. Overview of the course

Example 1.1. Let $x \in (0,1)\backslash\mathbb{Q}$ be an irrational number. It can be uniquely expressed as a continued fraction

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}}}$$

where $a_1, a_2, \ldots$ are positive integers.

How frequently does a positive integer k appear in such an expression?

It turns out that for any given k, the occurrence frequency of k in the continued fraction representation of x is the same for *almost every* $x \in (0,1)\backslash\mathbb{Q}$, and is given by the following formula

$$\lim_{n\to\infty} \frac{\#\{i \mid a_i = k, 1 \le i \le n\}}{n} = \frac{1}{\log 2} \log\left(\frac{(k+1)^2}{k(k+2)}\right).$$

To establish this result, we will introduce some fundamental concepts from *measure theory* and *ergodic theory.*

Example 1.2. Consider the following *necklace-splitting problem.* Two thieves have stolen a precious necklace (which is open, with two ends), adorned with d types of stones (such as diamonds, sapphires, rubies, etc.), with an even number of each type. Lacking knowledge of the values of the stones, the thieves aim to divide the stones of each type evenly using the fewest possible cuts. The question arises: what is the minimum number of cuts required to achieve this goal?

It is straightforward to demonstrate that at least d cuts are necessary: arrange the stones of each type successively, requiring one cut for each type. The *necklace theorem* establishes that d cuts are always sufficient. Remarkably, all known proofs of this theorem are *topological* in nature.

Example 1.3. Let $C \subseteq \mathbb{R}^2$ be a simple closed curve. One considers the following *Rectangular Peg Problems.*

- Is it always possible to find four points on C such that they form the vertices of a rectangle?
- A significantly more challenging question: Given a fixed rectangle R, is it always possible to find four points on C such that they form the vertices of a rectangle similar to R?

The first question was affirmatively answered by Vaughan in 1981, employing basic *topological* techniques. The second question was also recently resolved in the affirmative by Greene and Lobb [9]; however, their proof relies on more advanced tools from *symplectic geometry*, which fall beyond the scope of this course.

Example 1.4. For which positive integers n can we express n as the sum of two squares, i.e., $n = x^2 + y^2$?

To tackle this question, it is intuitive to introduce the *ring* of *Gaussian integers* $\mathbb{Z}[i]$, given the factorization $x^2 + y^2 = (x + iy)(x - iy)$. Consequently, the question is reduced to investigating the properties of the ring $\mathbb{Z}[i]$.

Example 1.5. A more refined question to consider is: how many ways can a positive integer n be expressed as the sum of two (or more) squares?

This problem is intricately connected to the *theta function*, defined for a complex variable $\tau \in \mathbb{H}$ on the upper half plane:

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 \tau} = \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \text{where} \quad q = \exp(2\pi i \tau).$$

Let $r_2(n)$ denote the number of ways n can be expressed as the sum of two squares:

$$r_2(n) = \#\{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = n\}.$$

It can be observed that

$$\theta(\tau)^2 = \sum_{n=0}^{\infty} r_2(n) q^n.$$

Thus, the problem reduces to understanding $\theta(\tau)^2$.

Remarkably, $\theta(\tau)^2$ is a *modular form of weight* 1 *for the congruence subgroup* $\Gamma_1(4) \subseteq \mathrm{SL}(2, \mathbb{Z})$. Utilizing the theory of modular forms allows us to derive an explicit formula for $r_2(n)$. Moreover, this approach extends to finding explicit formulas for $r_{2k}(n)$, representing the sum of $2k$ square numbers for any positive integer k, using modular forms.

Example 1.6. Is the rope depicted in the following figure knotted? Inspired by such questions, we will delve into various *knot invariants* and their *categorifications*, exploring the information they encapsulate. The process of categorification entails concepts such as *cobordism categories* and *topological quantum field theory*, which are of significant independent interest.

Example 1.7. In 1696, Johann Bernoulli posed the brachistochrone problem (from ancient Greek, meaning "shortest time") as a challenge to the mathematicians of his era: Given two points A and B in a plane, where B is lower but not directly below A, what is the curve traced out by a point acted upon solely by gravity, which starts from A and reaches B in the *shortest time*?

This problem is widely recognized as the seminal problem of the *calculus of variations*, which investigates methods for finding the curve or surface minimizing a given integral. We will explore the approaches developed by Euler (in 1736) and Lagrange (in 1755) for addressing general problems of this nature.

Example 1.8. Let $P = (p_1, \ldots, p_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function, where each coordinate $p_1, \ldots, p_n$ is a polynomial in $\mathbb{C}^n$. Independently proven by Grothendieck (1966) and Ax (1968), it was established that if P is injective, then it is bijective. This theorem can, in fact, be generalized to any *algebraic variety* over an *algebraically closed field*.

The method of proof is particularly noteworthy: it demonstrates the concept that *finitely many* algebraic relations in fields of characteristic 0 can be translated into algebraic relations over finite fields with large characteristics. Consequently, one can utilize the arithmetic of finite fields to prove a statement about $\mathbb{C}$, despite there being no homomorphism from any finite field to $\mathbb{C}$. This serves as an exemplary illustration of the application of techniques from *model theory* in *mathematical logic*.

Example 1.9. Let a and m be integers that are relatively prime. Is the sequence

$$a, a+m, a+2m, \ldots$$

infinitely populated by prime numbers?

This conjecture was initially proposed by Legendre and later proven by Dirichlet in 1837 using his *L-series*. This theorem is widely regarded as the cornerstone of rigorous *analytic number theory*. In fact, Dirichlet establishes a stronger result, demonstrating that the "density" of the subset

$$\{\text{prime } p \mid p \equiv a \pmod{m}\} \subseteq \{\text{prime } p\}$$

is $1/\varphi(m)$. In other words, prime numbers are equally distributed among different residue classes modulo m that are relatively prime to m.

Example 1.10. In 1657, Fermat corresponded with his friend de Bessy, his Dutch correspondent van Schooten, and English mathematicians Wallis and Brouncker, inviting them to tackle some intriguing mathematical problems. The central queries revolved around certain quadratic equations of the form

$$x^2 - Ny^2 = 1, \qquad x, y \in \mathbb{Z}_{>0}.$$

To Wallis and Brouncker, he presented challenges for the cases $N = 151$ and $N = 313$, while to his countryman de Bessy, he requested solutions for the cases $N = 61$ and $N = 109$, "so as not to give him too much trouble".

More broadly, this problem can be viewed as understanding the values of integral binary quadratic forms, such as $3x^2 + 6xy - 5y^2$. We will take a brief journey into the concept of *Conway's topograph*, featuring his *wells, rivers, lakes*, and *weirs*, to see how these aid us in addressing the problem.

Example 1.11. The dilogarithm function is defined by the power series

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \text{for } |z| < 1.$$

Its name and definition are inspired by the analogy with the Taylor series expansion of the ordinary logarithm around 1

$$-\log(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad \text{for } |z| < 1,$$

which leads to the definition of the *polylogarithm*

$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \qquad \text{for } |z| < 1, \qquad m = 1, 2, \ldots.$$

The dilogarithm function is one of the simplest non-elementary functions imaginable, yet it is also one of the most enigmatic. Its appearances in mathematics, along with the formulas associated with it, often possess a fantastical quality.

We will explore its connections with *hyperbolic* 3-*manifolds*, the *quantum dilogarithm identity*, and the *wall-crossing formula of stability conditions*.

Example 1.12. Consider the power series

$$\sum_{k=0}^{\infty}(-1)^k k! x^{k+1}.$$

It is evident that this series diverges for any $x \neq 0$, which might initially seem unremarkable. However, series of this form often arise naturally in various contexts. For example, they may represent solutions of ordinary differential equations or provide values for physical quantities such as energy.

Many mathematicians and physicists have recently taken interest in these series due to their prevalence in cutting-edge research topics, including gauge theory of singular connections, quantization of symplectic and Poisson manifolds, Floer homology and Fukaya categories, knot invariants, wall-crossing and stability conditions in algebraic geometry, perturbative expansions in quantum field theory, and more.

We will explore an approach to address the issue of divergence through the method of *Borel summation*. Along the way, we will encounter intriguing phenomena such as *resurgence*, the *Stokes phenomenon*, and relate them back to the *wall-crossing formula*.