

# GEOMETRY AND SYMMETRY, FALL 2022

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*Course Description.* Group theory is a fundamental mathematical concept that elucidates the symmetry of objects, finding applications across various fields of modern science. This course serves as an introduction to the basic principles and methods of group theory, utilizing examples and geometric phenomena to foster an intuitive comprehension of employing group methods in the study of symmetries.

The course commences with an exploration of the symmetries exhibited by common geometric objects in plane and solid geometry. It then progresses to introduce the definition and fundamental principles of groups, subsequently applying these concepts to topics such as symmetry groups of regular polyhedra, plane symmetry groups (wallpaper classification), Möbius transformations, and other related content and methodologies.

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